Yang-Mills Solutions on Manifolds with G-Structure

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Referent:	Prof. Dr. Olaf Lechtenfeld Leibniz Universität Hannover
Koreferent:	Prof. Dr. Roger Bielawski Leibniz Universität Hannover
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"Die Definition von Wahnsinn ist, immer wieder das Gleiche zu tun und andere Ergebnisse zu erwarten."

Albert Einstein

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Zusammenfassung

In dieser Arbeit untersuche ich höherdimensionale Instantonen und die BPS-Bedingung nicht erfüllende Yang-Mills-Lösungen auf Kegeln und Zylindern über kompakten G-Struktur-Mannigfaltigkeiten. Solche Mannigfaltigkeiten sind Bestandteil von heterotischen Stringkompaktifizierungen. Viele explizit bekannte G-Struktur-Mannigfaltigkeiten haben die Struktur von homogenen Räumen G/H. Ihre Kegel tauchen z. B. als Bran-Lösungen der heterotischen Stringtheorie und im Rahmen der AdS/CFT-Dualität auf.

In der heterotischen Supergravitation folgt die Existenz einer G-Struktur auf der kompakten Mannigfaltigkeit aus der Forderung, dass der Eichzusammenhang eine höherdimensionale Instantongleichung erfüllt. Explizite Instantonlösungen können Bausteine für Lösungen der heterotischen Supergravitation sein. Ich formuliere die Instantongleichung mit einem speziellen Ansatz für das Eichfeld auf dem Zylinder über einem allgemeinen homogenen Raum und erhalte dadurch eine Bedingung, die mit einem vereinfachten Ansatz gelöst werden kann. Die Lösungen haben Kinkform und sind bereits aus früheren Arbeiten bekannt. Eine Verallgemeinerung des Ansatzes führt auf Differentialgleichungen und algebraische Bedingungen, die auf einem beliebigen homogenen Raum nicht gelöst werden können. Ich betrachte daher das Beispiel $SU(3)/(U(1) \times U(1))$. Dieser Raum hat eine halbflache $SU(3)$-Struktur, die zu einer nearly-Kähler-Struktur reduziert werden kann. Im nearly-Kähler-Fall wurden Instantonen auf diesem Raum bereits in früheren Arbeiten untersucht. Ich schreibe die Instantongleichung für den halbflachen Fall auf. Mit dem Ergebnis lassen sich die bereits bekannten Lösungen reproduzieren und möglicherweise neue Instantonen konstruieren.

Ableiten der Instantongleichung führt auf die Yang-Mills-Gleichung mit einem Torsionsterm, der mit der total antisymmetrischen Torsion des Spin-Zusammenhangs aus der Supergravitation identifiziert werden kann. Die Yang-Mills-Gleichung ist die Bewegungsgleichung einer aus einem Yang-Mills- und einem Chern-Simons-Term bestehenden Wirkung. Ich untersuche die Yang-Mills-Gleichung mit total antisymmetrischer Torsion auf dem Kegel über einem allgemeinen homogenen Raum und konstruiere mit einem vereinfachten Ansatz für den Eichzusammenhang Lösungen, die Kinkform haben, nicht die BPS-Bedingung erfüllen und in ähnlicher Form in früheren Arbeiten vorkommen.

Kompakte Mannigfaltigkeiten mit Sasaki-Einstein-Struktur existieren in beliebiger ungerader Dimension und finden z. B. Anwendung in der AdS/CFT-Korrespondenz als supersymmetrische String-Hintergründe. Ich betrachte abschließend die Yang-Mills-Gleichung auf dem Zylinder über einer Sasaki-Mannigfaltigkeit und konstruiere neue analytische und numerische Instanton-, Dyon- und Sphaleron-Lösungen.

Abstract

In this thesis I study higher-dimensional instantons and non-BPS Yang-Mills solutions on cones and cylinders over compact G-structure manifolds. Such manifolds appear as internal spaces in heterotic string compactifications. Many explicitly known G-structure manifolds are homogeneous spaces of the form G/H. Their cones appear for example as brane solutions in heterotic string theory and in the context of the AdS/CFT duality.

In heterotic supergravity, the requirement that the compact manifold admit a G-structure follows from demanding the gauge connection to be an instanton. Explicit instanton solutions can serve as building blocks for heterotic supergravity solutions. I rewrite the higher-dimensional instanton condition on the cylinder over a general homogeneous space, using a special ansatz for the gauge field. The resulting conditions can be solved with a simplified ansatz, leading to kink-type solutions that are known from earlier works. A generalization yields differential and algebraic equations that cannot be solved in the general case. I therefore specialize to the cylinder over the coset space $SU(3)/(U(1) \times U(1))$. This space admits a half-flat $SU(3)$-structure which can be reduced to a nearly-Kähler structure. Instantons in the nearly-Kähler case have been studied in earlier work. I formulate the instanton equation in the half-flat case, obtaining a set of equations that allows for the reproduction of known solutions and may open the possibility for the construction of new instantons.

Differentiation of the instanton equation leads to the Yang-Mills equation with torsion. The torsion term can be identified with the totally antisymmetric torsion of the spin connection, naturally appearing in supergravity. The Yang-Mills equation extremizes an action consisting of a Yang-Mills and a Chern-Simons term. I consider the Yang-Mills equation with totally antisymmetric torsion on the cone over a general coset space and construct several non-BPS kink-type solutions with a simplified ansatz for the gauge connection that appear in a similar form in earlier work.

Compact manifolds with Sasaki-Einstein structure exist in any odd dimension and appear for example as supersymmetric string backgrounds in the context of the AdS/CFT correspondence. I finally specialize to the Yang-Mills equation on the cylinder over a Sasakian manifold and construct new analytic and numerical instanton, dyon and sphaleron solutions.

Contents

1 Introduction

Two major open problems of theoretical physics are the construction of a quantum theory of gravity and the unification of all four fundamental forces of nature in one consistent model, including the standard model of particle physics. String theory seems to be a promising candidate for solving both of these problems.

Phenomenologically viable models have to be consistently defined in four spacetime dimensions and should come with supersymmetry. Supersymmetric theories are attractive, as they suggest a solution of the hierarchy problem and a unification of the standard model coupling constants. String theories necessarily have to be supersymmetric in order to admit fermions in their particle spectrum. Requiring the preservation of the minimal ($\mathcal{N} = 1$) amount of supersymmetry implies that the variations of fermionic fields must vanish. This constraint leads to first-order BPS conditions that (in the cases of interest for us) imply the full second-order equations of motion of the theory. The first-order conditions are much easier to solve than the second-order equations. $\mathcal{N} = 1$ supersymmetry can be spontaneously broken at low energies, possibly explaining that supersymmetry has not been observed in experiments up to now.

A promising candidate for the construction of realistic models is the heterotic string [1]. This theory is supersymmetric and allows for a gauge connection with gauge group $SO(32)$ or $E_8 \times E_8$. Both groups admit an embedding of the standard model gauge group $SU(3) \times SU(2) \times U(1)$, and E_8 contains the GUT groups $SU(5)$, $SO(10)$ as well as E_6 as subgroups. The low-energy approximation of the heterotic string is heterotic supergravity, consisting of ten-dimensional $\mathcal{N} = 1$ supergravity coupled to ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory [2, 3]. Heterotic supergravity comes with an interesting gauge sector, naturally incorporates gravity and admits fermions in its particle spectrum. However, it is consistently defined in ten instead of four dimensions.

A common method for reducing the model to an effectively four-dimensional one is compactification. This idea, which originally appeared in the reduction of a five-dimensional theory to four dimensions in [4, 5], can be applied not only to the reduction of ten-dimensional heterotic string theory, but also to eleven-dimensional M- and twelve-dimensional F-theory, as well as to other higher-dimensional models. The idea is to split the originally D-dimensional spacetime into a direct product of a four-dimensional flat and a compact internal manifold of sufficiently small size to be unobservable in current experiments:

$$M_D = M_4 \times X_{D-4}. \tag{1.1}$$

In order to obtain a theory with $\mathcal{N} = 1$ supersymmetry in four dimensions, it turns out that the compactification space X_{D-4} must allow for at least one globally defined spinor ϵ that is covariantly constant with respect to the Levi-Civita connection ${}^{LC}\nabla$. This is equivalent to X_{D-4} having reduced holonomy of ${}^{LC}\nabla$. In the heterotic case, string spacetimes with four-dimensional $\mathcal{N} = 1$ supersymmetry can be constructed using a Calabi-Yau three-fold, i. e. a six-dimensional manifold with $SU(3)$-holonomy, as compactification space. However, compactifications of this kind give rise to a large number of so-called moduli, scalar fields with undetermined vacuum expectation value, in the effective theory.

A way out of the moduli problem that also relaxes the restriction on the background geometry is to allow for fluxes. These are nonvanishing form fields on the compact part of spacetime. For a review, see for example [6–8]. Fluxes first appeared in the context of heterotic string theory in [9]. In their presence, many moduli can be fixed and the compact manifold is no longer required to have reduced holonomy with respect to the Levi-Civita connection. Instead, X_{D-4} must admit a G-structure, i. e. a reduction of the tangent bundle structure group $SO(D-4)$ to some subgroup G. G-structure manifolds have reduced holonomy with respect to some generally torsionful connection. Flux compactifications do address the moduli problem but significantly enlarge the number of possible string backgrounds, which leads to the so-called string landscape problem.

The requirement that X_{D-4} admits a reduction of the structure group is closely related to the requirement that the gauge connection of ten-dimensional super-Yang-Mills theory satisfies a generalized instanton condition on the compact space-time part. Yang-Mills instantons in four dimensions [10] are classical solutions of the Yang-Mills equations of motion with finite, nonvanishing action. They constitute non-perturbative BPS configurations which satisfy the self-duality equation $*_4\mathcal{F} = \pm\mathcal{F}$, with $*_4$ denoting the Hodge star operator on the four-dimensional manifold. Covariant differentiation of the self-duality equation implies the Yang-Mills equation, hence instantons minimize the Yang-Mills action functional. Four-dimensional instantons have been intensely studied in the past decades. This has led to new insights both in mathematics and physics, such as for example a better understanding of Yang-Mills vacua and a classification of four-manifolds. For more details about instantons, see also [11, 12].

Generalizations of instantons to higher dimensions, first studied in [13, 14], are in particular interesting in the above described context for the construction of heterotic supergravity solutions. They also appear as brane solutions in the context of the AdS/CFT duality [15]. The generalization of the self-duality equation requires the existence of a globally defined four-form Q, which is equivalent to a reduction of the tangent bundle structure group. The generalized instanton equation is well-defined on any G-structure manifold and takes the form[1] [2]

$$*\mathcal{F} = -(*Q) \wedge F, \tag{1.2}$$

where $*$ now denotes the Hodge star operator on the higher-dimensional G-structure manifold. Applying a gauge covariant derivative leads to the Yang-Mills equation with torsion. It can be shown that higher-dimensional instantons extremize an action consisting of a Yang-Mills and a Chern-Simons term. As introduced in the four-dimensional case, higher-dimensional instantons are required to have finite action. In contrast to the four-dimensional case, explicit instanton solutions in higher dimensions are rare in the literature, and little is known about their moduli spaces. Explicit solutions have been constructed for instance on flat Euclidean space [16–22], as well as on cones and cylinders over general coset spaces [23–28]. This construction has been generalized to instantons on cones over real Killing spinor manifolds, and the constructed solutions have been lifted to new solutions of heterotic supergravity in [15, 29].

In the context of string compactifications, compact manifolds of dimensions five, six, seven and eight with reduced structure group and with special holonomy of the Levi-Civita connection are of particular interest. There is only a finite number of possible types of Levi-Civita holonomy groups on Riemannian manifolds, all of which have been listed by Berger in [30]. They include in particular Kähler, Calabi-Yau, G_2 and $Spin(7)$-manifolds. Today, explicit examples of compact manifolds are known for all listed cases. Apart from the appearance of

[1] More generally, higher-dimensional instantons can be defined as two-forms satisfying the condition $*\mathcal{F} = \nu(*Q) \wedge F$ for some real constant ν (see [29] for details).

[2] In later chapters, we will omit the parenthesis and write $*\mathcal{F} = - * Q \wedge \mathcal{F}$.

Calabi-Yau three-folds in heterotic compactifications, the seven-dimensional manifolds in Berger's list appear in the context of M-theory, and eight-dimensional $Spin(7)$-manifolds may be applied to compactifying M-theory to three dimensions, or to reducing F-theory from twelve to four dimensions. All listed spaces appear as target space geometries in supersymmetric sigma-models, and they are all related to Sasakian geometries [39]. The latter can be constructed in any odd dimension and hence appear as compactification spaces of various models. Sasakian manifolds that are in addition Einstein serve as building blocks of supersymmetric AdS/CFT solutions [33, 34].

A special class of G-structure manifolds are real Killing spinor manifolds. They are of particular interest for our work. It has been observed by Bär that the cone over a real Killing spinor manifold has special holonomy, allowing for a classification of G-structure manifolds [31]. All real Killing spinor manifolds admit a G-structure, but the converse is not generally true. Bär's list includes in particular nearly-Kähler, nearly-parallel G_2 and Sasaki-Einstein manifolds. All of them admit a connection with totally antisymmetric torsion that plays an important role in the construction of heterotic supergravity solutions. The torsion term of the Yang-Mills equation vanishes on manifolds with real Killing spinor.

Many known examples of G-structure manifolds are homogeneous spaces of the form G/H, where G is a compact Lie group and H a closed Lie subgroup. Product spaces of the form $\mathbb{R} \times G/H$ are of particular interest for the construction of instanton and Yang-Mills solutions. They can be considered as the simplest models of dimensional reduction of a D-dimensional theory to some $(D-1)$-dimensional theory, in which the fields depend only on one coordinate. As already mentioned, explicit instanton solutions have been constructed on cones and cylinders over coset spaces, for instance on the Euclidean spaces $\mathbb{R}^7$ and $\mathbb{R}^8$. These are the cones over the spheres S^6, S^7. They come with nearly-Kähler and nearly-parallel G_2-structure, respectively. Cones over coset spaces also appear as building blocks of certain string spacetimes, for example as brane solutions in heterotic supergravity [15] or as heterotic domain wall solutions [35].

Interesting solutions to the torsionful Yang-Mills equations that do not follow from a first-order equation and are therefore explicitly non-BPS can be constructed with an ansatz inspired by heterotic supergravity. The bosonic field content of heterotic supergravity [36, 37] is given by a metric (graviton) g_{AB}, a dilaton ϕ, a Kalb-Ramond two-form B and a gauge field $\mathcal{A}$ with gauge group either $SO(32)$ or $E_8 \times E_8$, the curvature of which is denoted by $\mathcal{F}$. In addition, we have a curvature three-form $\mathcal{H}$, which is obtained as the exterior derivative of B plus a combination of Chern-Simons forms [38]. The fermionic superpartners of the fields are the dilatino λ, the gaugino χ and the gravitino ψ. It turns out [38] that a particular connection is preferred in order to retain the equations of motion and first-order BPS conditions of heterotic supergravity in a simple form. Invariance of the supergravity action under ten-dimensional $\mathcal{N} = 1$ supersymmetry implies the following BPS equations (cf. [36, 37]):

$$\delta\psi = {}^{-}\nabla\epsilon = 0, \tag{1.3}$$

$$\delta\lambda = \gamma\left(d\phi - \frac{1}{2}\mathcal{H}\right)\cdot\epsilon = 0, \tag{1.4}$$

$$\delta\chi = \gamma(\mathcal{F})\cdot\epsilon = 0. \tag{1.5}$$

In these equations, $\epsilon \in \Gamma(\mathcal{S})$ is a Majorana-Weyl spinor, where $\mathcal{S}$ denotes the spinor bundle over the spacetime manifold M. γ is a map from k-forms to the Clifford algebra, which acts on the spinor ϵ via Clifford multiplication. The above mentioned preferred connection is ${}^{-}\nabla$, which has torsion proportional to the three-form $\mathcal{H}$. Its components are given by a combination of the Levi-Civita connection and the torsion term:

$${}^{-}\Gamma^{C}_{AB} = {}^{LC}\Gamma^{C}_{AB} + \frac{1}{2}\mathcal{H}^{C}_{AB}. \tag{1.6}$$

The torsion term of the higher-dimensional Yang-Mills equation may be chosen proportional to the torsion of this connection. For a certain proportionality factor, the torsionful Yang-Mills equation follows from the higher-dimensional instanton equation, as described above. In this case, solutions of the Yang-Mills equation can serve as building blocks of supersymmetric heterotic supergravity solutions. For other factors, the torsionful Yang-Mills equation does not follow from the instanton equation. The non-BPS solutions of this equation are candidates for building blocks of non-supersymmetric string solutions. This ansatz has been addressed for certain geometries in [23, 26] and is discussed in this thesis for cones over general coset spaces and cylinders over Sasakian manifolds.

Outline and Summary of Results

The outline of this thesis is as follows. We start by reviewing the geometry of homogeneous spaces in Chapter 2. In Chapter 3, we introduce gauge connections on G-structure manifolds and coset spaces. In Chapter 4, we describe details of special holonomy, G-structures and intrinsic torsion and discuss in particular manifolds with $SU(3)$-, G_2- and Sasakian structure. Instantons and the concept of self-duality in higher dimensions, as well as its meaning for the higher-dimensional torsionful Yang-Mills equation are discussed in Chapter 5.

In Chapter 6, we study the higher-dimensional instanton equation on product spaces of the form $\mathbb{R} \times G/H$. In this setup, the instanton equation splits into an algebraic condition and a first-order differential equation, which we write out explicitly. These conditions can be solved using a simplified ansatz for the gauge connection with one scalar function. We present an explicit kink-type solution on the cylinder over a general coset space. This solution already appeared in [23] in a similar context. With a more general ansatz for the gauge connection, we have to specialize to explicit examples of G/H. This is done in Chapter 7, where we choose to consider the half-flat $SU(3)$-structure manifold $SU(3)/(U(1) \times U(1))$. The product space $\mathbb{R} \times SU(3)/(U(1) \times U(1))$ admits a torsionful G_2-structure. Instantons on this product space have been studied before in [24, 26] under the assumption that the space $SU(3)/(U(1) \times U(1))$ be nearly-Kähler. We formulate the instanton equation on $\mathbb{R} \times SU(3)/(U(1) \times U(1))$, allowing the coset to have a half-flat structure. This yields a set of equations that include results from earlier works as special cases and may lead to new instanton solutions.

In Chapter 8, we study non-BPS solutions of the Yang-Mills equation on cones over general coset spaces, using a connection with totally antisymmetric torsion proportional to the three-form $\mathcal{H}$. The Yang-Mills equation is not conformally invariant, hence it has to be considered separately on the cone and on the cylinder. We derive the Yang-Mills equation on the cone over G/H, generalizing a result from [26], and solve this equation explicitly for the simplest possible gauge connection on the cone over G/H. This leads to various non-BPS kink-type solutions that are similar to solutions of earlier works (see e.g. [23]).

In Chapter 9, we turn to the cylinder $\mathbb{R} \times M$, where M has Sasakian structure and the gauge connection depends on two scalar functions. We study the torsionful Yang-Mills equation on this space in a similar way as in Chapter 8. Taking the product with a circle $S^1 \times M$ instead of $\mathbb{R} \times M$, we obtain periodic solutions with a sphaleron interpretation. Considering the product space $i\mathbb{R} \times G/H$ instead of $\mathbb{R} \times G/H$ leads to a sign flip in the potential. Solutions to this case are known as dyons. We recover the BPS solutions on $\mathbb{R} \times M$ derived in [29] and construct new analytic and numerical non-BPS Yang-Mills, as well as dyon and sphaleron solutions.

Part I

Gauge Theory and Geometry

2 Homogeneous Spaces

As motivated in the introduction, compact manifolds with reduced tangent bundle structure group $G \subset SO(d)$ are particularly interesting for string compactifications. Many known examples of these G-structure manifolds are homogeneous spaces. In this chapter, we review some basic facts about their geometry, following [40].

To introduce homogeneous spaces, we first need to define the action of a group. Here and in the following, we assume that all manifolds and vector spaces are finite-dimensional. The **left action** of a group G on a manifold M is a smooth map

$$\begin{aligned} L : G \times M &\to M \\ (g, m) &\mapsto g \cdot m, \end{aligned} \tag{2.1}$$

such that $e \cdot m = m$ for the unit element $e \in G$ and $ab \cdot m = a \cdot (b \cdot m)$ for all $a, b \in G,\ m \in M$. The right action of a group may be defined in an analogous way.

A group action is called **transitive** if any two elements $m, n \in M$ can be connected by a group element, i. e. for any $m, n \in M$ there exists an element $g \in G$ such that $g \cdot m = n$.

The set $G_m := \{g \in G | g \cdot m = m\}$ of group elements that leave a point $m \in M$ fixed is called **isotropy group** at m. The **orbit** of a point $m \in M$ is the set

$G \cdot m = \{g \cdot m | g \in G\}$.

Let G be a Lie group and $H \subset G$ a closed subgroup. The quotient

$$G/H = \{gH | g \in G\} \tag{2.2}$$

of left cosets of H in G admits a natural transitive G-action. There are now two equivalent ways to define a **homogeneous space**. First, a homogeneous space is a manifold M on which a Lie group G acts in a transitive way. Equivalently, it is a manifold of the form G/H with G being a Lie group and $H \subset G$ a closed subgroup.

Let us denote by $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras of the groups G and H. A homogeneous space G/H is called **reductive** if there exists a subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{m}$ is $Ad(H)$-invariant, i. e. $Ad(h)\mathfrak{m} \subset \mathfrak{m} \; \forall \; h \in H$. $Ad(H)$-invariance implies $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. If H is connected, the converse holds as well.

Homogeneous spaces are related to Riemannian spaces as follows. The **isometry group** on a Riemannian manifold (M, g) is defined to be the set $I(M)$ of maps that preserve the metric in the following sense:

$$I(M) := \{f : M \to M | g_m(X, Y) = g_{f(m)}(df_m(X), df_m(Y)) \\ \forall \; m \in M, \; X, Y \in T_m M\}. \tag{2.3}$$

This set is turned into a group by taking composition of functions as group operation. It can be shown that the isometry group is a Lie group. A **Riemannian homogeneous space** is defined as a Riemannian manifold on which the isometry group $I(M)$ acts in a transitive way. Such a space is isomorphic to the quotient G/H, where G denotes the isometry group $G = I(M)$ and H is the isotropy subgroup of a point.

A reductive Riemannian homogeneous space $M = G/H$ admits a G-invariant metric. Let $a \in G$ and denote by $L_a : M \to M, \; m \mapsto a \cdot m$, the diffeomorphism induced by left action. Then a metric g on M is **G-invariant** if L_a is an isometry, i. e.

$$g_{eH}(X, Y) = g_{L_a(eH)}(dL_a(X), dL_a(Y)) \qquad \forall \; a \in G, \; X, Y \in T_{eH}(G/H). \tag{2.4}$$

Here, $T_{eH}(G/H)$ denotes the tangent space at $eH \in G/H$ and e is the unit element of G. On a reductive homogeneous space, we have the identification $T_{eH}(G/H) \cong \mathfrak{m}$.

A particular G-invariant metric is induced by the Killing form. The **Killing form** of a Lie algebra $\mathfrak{g}$ is defined as the following symmetric, bilinear form:

$$\begin{aligned} B: \ & \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \\ & (X, Y) \mapsto \text{tr}\,(\text{ad}\,(X) \circ \text{ad}\,(Y)), \end{aligned} \tag{2.5}$$

where

$$\text{ad} \ : \ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \tag{2.6}$$

denotes the adjoint representation of the Lie algebra $\mathfrak{g}$. Given that $\mathfrak{g}$ is the Lie algebra of a Lie group G, the Killing form of the group G is understood to be the Killing form of the corresponding Lie algebra $\mathfrak{g}$. It can be shown that the Killing form of G is nondegenerate if and only if G is semisimple. If G is a compact, semisimple Lie group, then its Killing form is negative definite.

Let G/H be a homogeneous space, where G is a compact, semisimple Lie group. The negative of the Killing form gives rise to a left-invariant Riemannian metric on G. Left-invariant metrics on G are in one-to-one correspondence with scalar products on $\mathfrak{g}$. The scalar product induced by the Killing form gives rise to a reductive splitting $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. The restriction of the scalar product to the subspace $\mathfrak{m}$, $-B|_{\mathfrak{m}}$, then induces a G-invariant metric on G/H.

A special class of homogeneous spaces are the symmetric ones. A **Riemannian symmetric space** is a simply connected Riemannian manifold (S, g) with the following property: for any point $m \in S$ there exists an element $s \in I(S)$ of the isometry group, called a symmetry at m, such that

$$s(m) = m, \qquad (ds)|_m = -id. \tag{2.7}$$

This implies in particular that the isometry group acts transitively, i. e. that S is a homogeneous space. We may therefore define a symmetric space to be a homogeneous space that admits a symmetry s at some point $m \in S$ and identify S with a coset space G/H. It can be shown that the holonomy group of the Levi-Civita connection on a symmetric space is contained in its isotropy group H. This fact will be of interest in the classification of holonomy groups in Chapter 4.

For explicit computations on coset spaces G/H, it is useful to choose a basis of generators $\{I_{\widetilde{a}}\}$ of $\mathfrak{g}$, where $\widetilde{a} = (1, \ldots, \dim \mathfrak{g})$. This basis can be used to construct a basis of one-forms $\{e^a\}$ on G/H as follows. The generators $I_{\widetilde{a}}$ of $\mathfrak{g}$ can be represented by left-invariant vector fields $\widehat{E}_{\widetilde{a}}$ on G. The dual basis of these vector fields is a set of left-invariant one-forms, denoted $\widehat{e}^{\widetilde{a}}$. We consider the coset space as a principal bundle $G \longrightarrow G/H$ and denote the natural projection that sends elements g of G to the corresponding coset gH by

$$\begin{aligned} \pi : G &\to G/H \\ g &\mapsto gH. \end{aligned} \tag{2.8}$$

Let us consider a small contractible open subset $U \subset G/H$ and choose a local section $\sigma : U \to \pi^{-1}(U) \subset G$ in the principal bundle, such that $\pi \circ \sigma = id$. The pullback of the left-invariant one-forms $\widehat{e}^{\widetilde{a}}$ by σ is denoted $e^{\widetilde{a}}$. These one-forms split into the sets $\{e^a\}$ and $\{e^i\}$, where $\{e^a\}$ constitutes an orthonormal frame of the dual tangent bundle $T^*(G/H)$ over U, and the elements e^i can be written as linear combinations $e^i = e^i_a e^a$ with real functions e^i_a. We denote by $\{E_a\}$ the local frame dual to $\{e^a\}$ on the tangent bundle $T(G/H)$. These frames can be transported outside of U by group action. In the same way as the one-forms, the generators of G split into two sets $\{I_a\}$ and $\{I_i\}$, where indices $i, j = ((\dim G - \dim H + 1), \ldots, \dim G)$ label the generators of H, and indices $a, b = (1, \ldots, (\dim G - \dim H))$ label the generators $\{I_a\}$, spanning the subspace $\mathfrak{m}$ of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ is characterized by the structure constants $f^{\widetilde{c}}_{\widetilde{a}\widetilde{b}}$, which are defined via the commutation relations in the chosen basis:

$$[I_{\widetilde{a}}, I_{\widetilde{b}}] = f^{\widetilde{c}}_{\widetilde{a}\widetilde{b}} I_{\widetilde{c}}. \tag{2.9}$$

Taking the splitting $\widetilde{a} = (a, i)$ of indices into account, the commutation relations take the form

$$[I_i, I_j] = f^k_{ij} I_k, \tag{2.10}$$
$$[I_i, I_a] = f^b_{ia} I_b + f^k_{ia} I_k, \tag{2.11}$$
$$[I_a, I_b] = f^c_{ab} I_c + f^k_{ab} I_k. \tag{2.12}$$

Structure constants of the form f^a_{ij} vanish, as H is closed. On a reductive homogeneous space, we find

$$[I_i, I_j] = f^k_{ij} I_k, \tag{2.13}$$
$$[I_i, I_a] = f^b_{ia} I_b, \tag{2.14}$$
$$[I_a, I_b] = f^c_{ab} I_c + f^k_{ab} I_k, \tag{2.15}$$

due to $Ad(H)$-invariance of $\mathfrak{m}$. On symmetric spaces, we have in addition $f^c_{ab} = 0$.

It can be shown that the one-forms constructed above satisfy the Maurer-

Cartan equations, which read as follows using the shorthand notation $e^a \wedge e^b := e^{ab}$:

$$de^a = -\frac{1}{2} f^a_{\widetilde{b}\widetilde{c}} e^{\widetilde{b}\widetilde{c}} = -\frac{1}{2} f^a_{bc} e^{bc} - f^a_{ic} e^{ic}, \tag{2.16}$$

$$de^i = -\frac{1}{2} f^i_{\widetilde{b}\widetilde{c}} e^{\widetilde{b}\widetilde{c}} = -\frac{1}{2} f^i_{bc} e^{bc} - f^i_{jk} e^{jk}. \tag{2.17}$$

The metric induced by the Killing form on G can be written in terms of the structure constants as

$$(g_K)_{\widetilde{a}\widetilde{b}} = -\operatorname{tr}\left(\operatorname{ad}\left(I_{\widetilde{a}}\right) \circ \operatorname{ad}\left(I_{\widetilde{b}}\right)\right) = f^{\widetilde{c}}_{\widetilde{a}\widetilde{d}} f^{\widetilde{d}}_{\widetilde{c}\widetilde{b}}. \tag{2.18}$$

We may choose the generators in such a way that the metric becomes

$$(g_K)_{\widetilde{a}\widetilde{b}} = f^{\widetilde{c}}_{\widetilde{a}\widetilde{d}} f^{\widetilde{d}}_{\widetilde{c}\widetilde{b}} = \delta_{\widetilde{a}\widetilde{b}}. \tag{2.19}$$

On a reductive homogeneous space, this metric decomposes further as

$$(g_K)_{ab} = 2 f^i_{ad} f^d_{ib} + f^c_{ad} f^d_{cb} = \delta_{ab}, \tag{2.20}$$

$$(g_K)_{ij} = 2 f^k_{il} f^l_{kj} + f^b_{ia} f^a_{bj} = \delta_{ij}, \tag{2.21}$$

$$(g_K)_{ai} = 0. \tag{2.22}$$

We will restrict our consideration to coset spaces that satisfy the following relations, where $\alpha \in \mathbb{R}$ is a parameter specific for a chosen coset space:

$$f^c_{ad} f^d_{cb} = \alpha \delta_{ab}, \tag{2.23}$$

$$f^i_{ad} f^d_{ib} = \frac{1}{2}(1-\alpha)\delta_{ab}, \tag{2.24}$$

$$f^c_{ad} f^d_{ci} = \delta_{ai} = 0. \tag{2.25}$$

These relations do not hold for arbitrary coset spaces, but they are satisfied for most of the spaces that are relevant for us. Written in a basis, the G-invariance condition of the metric turns into the following constraint, with respect to the above splitting of indices:

$$f^c_{i(a} g_{b)c} = 0. \tag{2.26}$$

2.1 Cones and Cylinders

Product spaces of the form $\mathbb{R} \times G/H$ are of particular interest for the construction of instantons and non-BPS Yang-Mills solutions. Warped geometries, in which the metric is not the standard product one, have various applications in string compactifications. In this work, we are particularly interested in cones and cylinders. Let (M, g_M) be a Riemannian manifold. We define

1. the **cylinder** over M as the product space $\mathcal{Z}(M) = (\mathbb{R} \times M, g_\mathcal{Z})$ with metric $g_\mathcal{Z} = d\tau^2 + g_M$,
2. the **Riemannian** or **metric cone** over M as the warped product $\mathcal{C}(M) = (\mathbb{R} \times M, g_\mathcal{C})$ with metric $g_\mathcal{C} = dr^2 + r^2\gamma^2 g_M$, where r^2 is also known as **warping function** and γ^2 denotes the **opening angle**.

These two metrics are conformally equivalent, which can be seen by introducing the relation $r := e^{\gamma\tau}$. Then the cone metric takes the following form, which differs from the cylinder metric only by a conformal factor $\gamma^2 e^{2\gamma\tau}$:

$$g_\mathcal{C} = \gamma^2 e^{2\gamma\tau}(d\tau^2 + g_M) = \gamma^2 e^{2\gamma\tau} g_\mathcal{Z}. \tag{2.27}$$

In the following chapters, we will use the variable τ both for the cone and for the cylinder metric.

3 Connections in Tangent and Principal Bundles

For the introduction of G-structure and special holonomy, we need to understand the notion of connections in tangent and principal bundles. Assuming that the reader is familiar with the formulation of gauge theory in the language of bundles, we will focus on the quantities that are relevant for the work at hand, in particular for the explicit computations presented in Parts II and III of this thesis. More detailed discussions can be found in [41, 42]. For more information about bundles and connections, we also refer to [43].

Let $P(M, G)$ be a principal bundle with structure group G over a d-dimensional Riemannian manifold (M, g) and denote by $\pi : P \to M$ the projection onto the base space. Let $\rho : G \to GL(V)$ be a representation and $E = P \times_\rho V$ the corresponding associated vector bundle. The Lie algebra of the Lie group G is referred to as $\mathfrak{g}$. The space of $\mathbb{R}$-valued k-forms on M will be denoted $\Omega^k(M) := \Gamma(\wedge^k(T^*M))$, and the space of k-forms on M that take values in the vector space V is referred to as $\Omega^k(M, V) := \Gamma(\wedge^k(T^*M) \otimes V)$.

The principal bundle $P(M, G)$ is endowed with a $\mathfrak{g}$-valued connection one-form that determines a splitting of the tangent bundle of P into a horizontal and a vertical subbundle. In more detail, let $X \in \mathfrak{g}$ and $u \in P$. Then $u \cdot \exp(tX)$, with $t \in \mathbb{R}$, is a curve in P through the point u. This curve lies within the fiber G_p at $p = \pi(u) \in M$. We can now define a vector at the point $u \in P$ as

$$\widetilde{X}(u) := \frac{d}{dt}(u \cdot \exp(tX))|_{t=0}. \tag{3.1}$$

The corresponding vector field $\widetilde{X} \in \Gamma(TP)$ is referred to as **fundamental vector field**. By construction, $\widetilde{X}(u)$ is tangent to G_p. We therefore have $\widetilde{X}(u) \in V_uP \subset T_uP$, where $V_uP \cong \mathfrak{g}$ denotes the subspace of T_uP tangent to G_p. We refer to V_uP

as **vertical subspace**.

A **connection** in $P(M,G)$ is a $\mathfrak{g}$-valued one-form $A \in \Omega^1(P,\mathfrak{g})$ that satisfies

$$A(\widetilde{X}) = X \;\forall\; X \in \mathfrak{g} \qquad \text{and} \qquad R_g^* A = Ad_{g^{-1}} \circ A, \tag{3.2}$$

where $g \in G$, R_g denotes right multiplication by g and $Ad_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ is the adjoint map.

The connection one-form can be understood as a projection of T_uP onto the vertical subspace V_uP at every point $u \in P$. It therefore defines a splitting $T_uP = V_uP \oplus H_uP$, where H_uP is referred to as **horizontal subspace**.

For explicit computations, it is convenient to use a local expression for the connection form A. Let therefore $U \subset M$ be a small open subset and $\sigma : U \to \pi^{-1}(U) \subset P$ a local section in P. A local $\mathfrak{g}$-valued one-form on U can then be constructed via pullback of the connection form:

$$\mathcal{A} = \sigma^* A \in \Omega^1(U,\mathfrak{g}). \tag{3.3}$$

We will also refer to this local form as **gauge connection**[3]. The local connection form $\mathcal{A}$ can be written as follows after introducing a local frame $\{e^A\}$ of T^*M and generators $\{I_A\}$ of $\mathfrak{g}$, with $\mathcal{A}_A^B$ denoting real functions:

$$\mathcal{A} = \mathcal{A}_A^B e^A \otimes I_B. \tag{3.4}$$

Note that any r-form $\phi \in \Omega^r(P,V)$ with values in a vector space V can be written as $\phi = \phi^A \otimes E_A$, with $\phi^A \in \Omega^r(P)$ being real-valued forms and $\{E_A\}, A = (1,\ldots,\dim(V))$, denoting a basis of V.

The curvature of a global connection form A can be introduced as follows. Denoting the decomposition of a vector $X \in T_uP$ into its horizontal and vertical components by $X = X^H + X^V \in H_uP \oplus V_uP$, the **exterior covariant derivative** $D : \Omega^r(P,V) \to \Omega^{r+1}(P,V)$ of an r-form ϕ is defined as

$$D\phi(X_1,\ldots,X_{r+1}) := d_P\phi(X_1^H,\ldots,X_{r+1}^H), \tag{3.5}$$

where $d_P\phi = (d_P\phi^A) \otimes E_A$ denotes the exterior derivative of an $\mathbb{R}$-valued differential form in the bundle $P(M,G)$.

We can now introduce the **curvature** of A as the $\mathfrak{g}$-valued two-form

$$F = DA \in \Omega^2(P,\mathfrak{g}). \tag{3.6}$$

[3]Note that the local expression depends on the section σ. The choice of a section corresponds to a choice of gauge.

The commutator of two $\mathfrak{g}$-valued differential forms $\eta = \eta^A \otimes I_A \in \Omega^p(P, \mathfrak{g})$ and $\omega = \omega^B \otimes I_B \in \Omega^q(P, \mathfrak{g})$ is defined as

$$[\eta, \omega] = \eta^A \wedge \omega^B \otimes [I_A, I_B]. \tag{3.7}$$

In particular, the commutator of a one-form $\xi \in \Omega^1(P, \mathfrak{g})$ satisfies

$$[\xi, \xi](X, Y) = 2[\xi(X), \xi(Y)] \tag{3.8}$$

for any two vector fields $X, Y \in \Gamma(TP)$. It can then be shown that the curvature satisfies the following identity, which is also referred to as Cartan's structure equation:

$$\begin{aligned} F(X, Y) &= (d_P A)(X, Y) + [A(X), A(Y)] \\ &= \left(d_P A + \frac{1}{2}[A, A] \right)(X, Y) \qquad \forall\ X, Y \in \Gamma(TP). \end{aligned} \tag{3.9}$$

In addition, F satisfies the Bianchi identity

$$DF = 0. \tag{3.10}$$

Again, we are interested in a local expression of the curvature. With the same notation as above, a local curvature form is constructed as

$$\mathcal{F} := \sigma^* F \in \Omega^2(U, \mathfrak{g}). \tag{3.11}$$

The local curvature form is related to the local connection form as

$$\mathcal{F}(X, Y) = (d\mathcal{A})(X, Y) + [\mathcal{A}(X), \mathcal{A}(Y)] \qquad \forall\ X, Y \in \Gamma(TM). \tag{3.12}$$

In this context, d denotes the exterior derivative of an $\mathbb{R}$-valued differential form on the base manifold M. This identity suggests to introduce a derivative

$$\begin{aligned} \mathcal{D} :\ \Omega^r(M, \mathfrak{g}) &\to \Omega^{r+1}(M, \mathfrak{g}) \\ \eta &\mapsto d\eta + [\mathcal{A}, \eta]. \end{aligned} \tag{3.13}$$

Using $\mathcal{D}$, equation (3.12) can be written as

$$\mathcal{F} = \mathcal{D}\mathcal{A}, \tag{3.14}$$

and the Bianchi identity takes the local form

$$\mathcal{D}\mathcal{F} = 0. \tag{3.15}$$

The gauge connection $\mathcal{A}$ in a principal bundle $P(M, G)$ induces a covariant derivative in any associated vector bundle $E = P \times_\rho V$. We will also use the notation $E(M, V)$ for vector bundles over M whose fibers are vector spaces V. A **covariant derivative** in a vector bundle $E(M, V)$ is a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \tag{3.16}$$

that satisfies the Leibniz rule

$$\nabla(fs) = df \otimes s + f \cdot \nabla s \qquad \forall\ f \in C^\infty(M),\ s \in \Gamma(E). \tag{3.17}$$

It can be expressed in local coordinates by a matrix-valued one-form $\Gamma_A^B = \Gamma_{CA}^B e^C$, where $\{e^A\}$ is a local frame of T^*M. The components Γ_{CA}^B are also referred to as connection coefficients, or as Christoffel symbols if ∇ is the Levi-Civita connection. The connection ∇ acts on one-forms $\eta = \eta_A e^A \in \Omega^1(M)$ with $\eta_A \in C^\infty(M)$ as

$$\nabla\eta = \left(d\eta_A - \eta_B \Gamma_A^B\right) \otimes e^A. \tag{3.18}$$

Of particular interest for our work is the tangent bundle TM, which is associated to the frame bundle

$$F(M, GL(d)) := \{(p, E_1, \ldots, E_d) | p \in M \text{ and } (E_1, \ldots, E_d) \text{ is a basis of } T_pM\}. \tag{3.19}$$

The frame bundle is a principal bundle with fiber $GL(d)$, and the associated tangent bundle has structure group $GL(d)$. We will see in Chapter 4.2 that the tangent bundle of a G-structure manifold has reduced structure group $G \subset GL(d)$. We have $TM = F \times_\rho \mathbb{R}^d$, where ρ is the standard $(d \times d)$-matrix representation of $GL(d)$. A connection A in the frame bundle is a map $A : TF \to \mathfrak{gl}(d)$ from TF to the real $(d \times d)$-matrices. In this case, the corresponding local connection form (3.4) can be written as

$$\mathcal{A}_M^N = \mathcal{A}_A^B e^A \otimes (I_B)_M^N \in \Omega^1(M, \mathfrak{gl}(d)), \tag{3.20}$$

where the $GL(d)$-generators $\{I_B\}$ are expressed as $(d \times d)$-matrices. Then the components

$$\mathcal{A}_{AM}^N = \mathcal{A}_A^B (I_B)_M^N \tag{3.21}$$

are the connection coefficients of the covariant derivative induced by $\mathcal{A}$ in TM.

Note that, if $M = G/H$ is a reductive homogeneous space, the tangent bundle is a subbundle of the adjoint bundle

$$Ad_P = P \times_{Ad(G)} \mathfrak{g}, \tag{3.22}$$

constructed by use of the adjoint representation,

$$Ad : G \to GL(\mathfrak{g}). \tag{3.23}$$

This is due to the splitting $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of Lie algebras and the identification $\mathfrak{g} \cong T_eG, \mathfrak{m} \cong T_{eH}(G/H)$, where e denotes the unit element in G.

Given a representation ρ of G, there is a one-to-one correspondence of principal bundles and associated vector bundles (for the explicit construction, see e. g. [42, 44]). As illustrated for the frame bundle, a connection in any principal bundle uniquely determines a covariant derivative in the associated vector bundle. It can be shown that the converse holds as well [44], hence connections in $P(M,G)$ and covariant derivatives in associated vector bundles are in one-to-one correspondence. We will therefore not explicitly distinguish between connections and covariant derivatives anymore in the following chapters.

Certain connections in the tangent bundle over the manifold (M, g) will play a special role in the following. The coefficients of any metric-compatible torsionful connection Γ on TM are uniquely determined by the conditions

$$dg_{AB} - g_{AC}\Gamma^C{}_B - g_{BC}\Gamma^C{}_A = 0, \tag{3.24}$$

$$de^A + \Gamma^A_{BC} e^B \wedge e^C = T^A, \tag{3.25}$$

where $T^A = \frac{1}{2} T^A_{BC} e^B \wedge e^C$ denotes the torsion two-form. Motivated by its appearance in heterotic supergravity, we introduce the **torsionful spin connection** ${}^{-}\Gamma$ with components

$${}^{-}\Gamma^N_{AM} = {}^{LC}\Gamma^N_{AM} + T^N_{AM} \tag{3.26}$$

as a metric-compatible connection with totally antisymmetric torsion. This connection will be used for the construction of Yang-Mills solutions in Chapter 9, where its coefficients are explicitly computed and the torsion is chosen to be proportional to the structure constants, $T_{abc} \propto f_{abc}$.

Furthermore, we can introduce a **canonical connection** in TM if M admits a reduction of the structure group. According to [29], the canonical connection ${}^{P}\Gamma$ in the tangent bundle TM over a G-structure manifold (M, g) is the unique connection whose holonomy[4] is equivalent to the reduced structure group and whose torsion is totally antisymmetric with respect to some G-compatible metric. In explicit examples, the torsion will be proportional to the G-structure three-

[4] cf. Chapter 4

form[5] P, i. e. $T_{AMN} \propto P_{AMN}$. The particular feature of this connection is that it satisfies the instanton equation.

The canonical connection takes a particularly simple form if $M = G/H$ is a reductive homogeneous space. The space G/H can be written as a principal bundle with structure group H and left action of G, using the natural projection $\pi : G \to G/H$. In this bundle, we find a unique G-invariant connection ${}^P\mathcal{A}$ [43, 45], which takes values in the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ and can locally be written as

$$ {}^P\mathcal{A} = e^i \otimes I_i \in \Omega^1(U, \mathfrak{h}), \tag{3.27} $$

where $U \subset G/H$ and the index i labels $\mathfrak{h}$-directions as in Chapter 2. The connection induced by ${}^P\mathcal{A}$ in $T(G/H)$ has components

$$ {}^P\mathcal{A}^N_{aM} = \mathcal{A}^i_a \otimes (I_i)^N_M = e^i_a \otimes (I_i)^N_M. \tag{3.28} $$

Using data from [29], the torsion of this connection can be explicitly computed in the cases where G/H has structure group $SU(3)$, G_2, $Spin(7)$ or $SU(m)$ in dimension $(2m+1)$. In these cases, it can be shown that ${}^P\Gamma$ and ${}^P\mathcal{A}$ have the same torsion, hence the connection induced by ${}^P\mathcal{A}$ is the same as ${}^P\Gamma$ on $T(G/H)$.

According to [43], G-invariant connections in the principal bundle $P(G/H, G)$ on a reductive homogeneous space G/H with values in the full Lie algebra $\mathfrak{g}$ (not only in the subalgebra $\mathfrak{h}$) are determined by linear maps $\Lambda : \mathfrak{m} \to \mathfrak{g}$ which commute with the adjoint action of H:

$$ \Lambda(Ad(h)Y) = Ad(h)\Lambda(Y) \qquad \forall\ h \in H,\ Y \in \mathfrak{m}. \tag{3.29} $$

In a basis $\{I_B\}$ of $\mathfrak{g}$-generators, such a linear map is represented by a matrix X^B_a as

$$ X_a := \Lambda(I_a) = X^B_a I_B = X^i_a I_i + X^b_a I_b \in \mathfrak{g}. \tag{3.30} $$

For the cases of interest, one can always choose $X^i_a = 0$, i. e. $X_a = X^b_a I_b \in \mathfrak{m} \subset \mathfrak{g}$. The connection takes the local form

$$ \mathcal{A} = e^a \otimes X_a = X^b_a e^a \otimes I_b \in \Omega^1(U, \mathfrak{m}) \tag{3.31} $$

in $U \subset G/H$. A G-invariant connection with values in the full Lie algebra $\mathfrak{g}$ is then given as a combination of the canonical connection with (3.31) as

$$ \mathcal{A} = e^i \otimes I_i + e^a \otimes X_a \in \Omega^1(U, \mathfrak{g}). \tag{3.32} $$

[5] The structure three-form will be introduced in Chapter 4.3 for Sasakian manifolds, in Chapter 4.4 for $SU(3)$-structure manifolds and in Chapter 4.5 for G_2-structure manifolds. In the $SU(3)$-structure case, it is denoted Ω instead of P.

G-invariance (3.29) requires the connection to satisfy the additional condition

$$[I_i, X_a] = f^b_{ia} X_b. \tag{3.33}$$

In the following chapters, we will study instanton and Yang-Mills solutions on spaces of the form $\mathbb{R} \times G/H$. The G-invariant connection on G/H lifts to a G-invariant connection on this product space, where the coefficients X^b_a turn into functions $X^b_a(\tau)$ on the $\mathbb{R}$-coordinate τ. A local frame on $\mathbb{R} \times G/H$ is given by $\{e^0, e^a\}$, where $e^0 := d\tau$ is a one-form on $\mathbb{R}$. We may choose the component $\mathcal{A}_0$ in $\mathbb{R}$-direction to vanish (temporal gauge) and compute the following curvature components:

$$\mathcal{F}_{0a} = \dot{X}_a, \tag{3.34}$$

$$\mathcal{F}_{bc} = -(f^i_{bc} I_i + f^a_{bc} X_a - [X_b, X_c]), \tag{3.35}$$

with $\dot{X}_a := \frac{\partial X_a(\tau)}{\partial \tau}$. The size and explicit shape of the matrices X^b_a depend on the chosen coset space and the structure of the representation of the generators I_A. Solutions to the G-invariance condition (3.33) have been constructed for certain coset spaces in [26]. We will use the result for $G/H = SU(3)/(U(1) \times U(1))$ in Chapter 7 (cf. equation (7.6)).

4 Manifolds with G-Structure and Special Holonomy

The requirement to recover an effectively four-dimensional theory with $\mathcal{N} = 1$ supersymmetry from string compactification imposes a condition on the geometry of the compact internal manifold. This condition can be conveniently formulated by use of spinors. Let (M, g) be a Riemannian spin manifold and $\mathcal{S}$ the spinor bundle over M. A **real Killing spinor** is a section $\epsilon \in \Gamma(\mathcal{S})$ that satisfies the equation

$$^{LC}\nabla\epsilon = \lambda\gamma \cdot \epsilon, \tag{4.1}$$

where $^{LC}\nabla$ is the Levi-Civita connection, λ a real constant and γ a representation of the Clifford algebra.

In the absence of fluxes, the preservation of $\mathcal{N} = 1$ supersymmetry requires that the compact manifold admits a nowhere vanishing spinor that satisfies $^{LC}\nabla\epsilon = 0$. This condition is equivalent to the requirement of special, reduced holonomy with respect to the Levi-Civita connection. The condition that the spinor on the compact manifold be covariantly constant can be relaxed when fluxes, i. e. nonvanishing three-forms, are present on M. In this case, the manifold need not necessarily have reduced holonomy of $^{LC}\nabla$ but must still be equipped with a nowhere vanishing real Killing spinor. This implies a reduction of the tangent bundle structure group $SO(n)$ to some subgroup G. In this chapter, we introduce both the concepts of holonomy and G-structure in greater detail.

Our focus will not be on the spinor approach in the following, as we use alternative characterizations of G-structures instead. We will only briefly mention spinors at certain points. For further details on the formulation of the geometric conditions in terms of spinors and their correspondence to the instanton equation, we refer to [29].

4.1 Special Holonomy

Let $E(M,V)$ be a vector bundle of rank d over a smooth manifold M, endowed with a connection ∇. Let $\gamma : [0,1] \to M$ be a piecewise smooth loop in M, starting and ending at the point $x \in M$. The connection defines the **parallel transport** of a section in E along such a path γ. A section $s \in \Gamma(E)$ is parallel along γ if its covariant derivative vanishes along the path. Then the parallel transport of a vector E_0 at $p = \gamma(0)$ is the unique section s that satisfies

$$\nabla_{\dot{\gamma}(t)} s = 0 \ \forall \ t \in [0,1] \qquad \text{and} \qquad s_{\gamma(0)} = E_0, \tag{4.2}$$

where $\dot{\gamma}(t) = \frac{\partial \gamma(t)}{\partial t}$. We can introduce the parallel transport map $\mathcal{P}_\gamma : V_x \to V_x$, which takes elements of a fiber V_x to elements in the same fiber by parallel transporting them along γ. Clearly, this map depends on the chosen curve and on the connection. Being linear and invertible, it defines a group element of $GL(d)$, acting on elements of the fiber V_x.

The **holonomy group** of a connection ∇ in a vector bundle $E(M,V)$ at a point $x \in M$ is defined to be the group of parallel translations along all closed loops starting and ending at x:

$$Hol_x(\nabla) := \{\mathcal{P}_\gamma \in GL(d) | \gamma \text{ is loop based at } x\} \subset GL(d). \tag{4.3}$$

It can be shown that if the base manifold M is connected, the holonomy group depends on the base point x only up to conjugation in $GL(d)$. We therefore write $Hol(\nabla)$ from now on. The **restricted holonomy group** at x is defined as the subgroup $Hol^0(\nabla)$ generated by all contractible loops γ. It is identical to $Hol(\nabla)$ if M is simply connected. As the notion of parallel transport is determined by the choice of connection, the holonomy group depends on the connection as well.

Let (M,g) be an oriented Riemannian manifold and denote by $Hol(g)$ the holonomy group of the Levi-Civita connection ${}^{LC}\nabla$, which is uniquely determined by the metric g. There is only a finite number of possible types of holonomy groups with respect to ${}^{LC}\nabla$ on oriented compact Riemannian manifolds. All of them have been listed by Berger in [30] and are also referred to as special holonomy groups. This list originally included $Spin(9)$-manifolds, but it was shown later [51] that all compact manifolds with $Spin(9)$-structure are symmetric. Symmetric spaces G/H are excluded in the list, as their holonomy group is known to be contained in H (cf. Chapter 2).

The statement of Berger's central theorem is as follows: Let (M,g) be an oriented Riemannian manifold which is neither locally a Riemannian product nor

$Hol^0(g)$	$\dim(M)$	Geometry of M
$SO(n)$	n	Orientable Riemannian
$U(n)$	$2n$	Kähler
$SU(n)$	$2n$	Calabi-Yau
$Sp(n) \cdot Sp(1)$	$4n$	Quaternionic Kähler
$Sp(n)$	$4n$	Hyperkähler
G_2	7	G_2-manifold
$Spin(7)$	8	$Spin(7)$-manifold

Table 1: Holonomy groups $Hol^0(g)$ on compact oriented Riemannian manifolds according to Berger.

locally symmetric. Then the restricted holonomy group $Hol^0(g)$ of the Levi-Civita connection is one of the groups listed in Table 1.

4.2 G-Structure and Intrinsic Torsion

Let us now turn to the related concept of G-structure manifolds. According to [44], we have the following definition: Let M be a d-dimensional manifold and $F(M, GL(d))$ the frame bundle. F is a principal bundle with structure group $GL(d)$. Let $G \subset GL(d)$ be a Lie subgroup. Then a G-**structure** is a principal subbundle $P(M, G)$ of F with fiber G.

As an example, consider a d-dimensional Riemannian manifold (M, g) and its frame bundle F with elements $(p, E_1, \ldots, E_d)$, where $p \in M$ denotes the base point and $\{(E_1, \ldots, E_d)\}$ are bases of T_pM. As the manifold is equipped with a metric, we may define the subset

$$P := \{(p, E_1, \ldots, E_d) \in F | (E_1, \ldots, E_d) \text{ orthonormal}\} \subset F. \tag{4.4}$$

P is a principal subbundle of F with fiber $O(d) \subset GL(d)$, and it fixes an $O(d)$-structure on M. If the basis $(E_1, \ldots, E_d)$ is in addition oriented, the structure group is $SO(d)$. In fact, there is a one-to-one correspondence of Riemannian metrics and $O(d)$-structures on M. In other words, a G-structure with $G \subset O(d)$ determines a metric on the manifold M. To avoid complications, we will restrict the following discussion to G-structures with $G \subset SO(d)$. Details on the more general case $G \subset GL(d)$ can be found for example in [49].

We are interested in connections in the tangent bundle of a G-structure manifold. Connections in the tangent bundle are related to connections in the G-subbundle $P(M, G)$ of the frame bundle $F(M, GL(d))$ as follows [44]: a connection ∇ in TM is called a **G-connection**, or **compatible with a given G-structure**, if the associated $\mathfrak{gl}(d)$-valued connection in the frame bundle reduces to a connection in P, i. e. if it takes values in the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(d)$.

As the structure group $G \subset SO(d)$ determines a metric on the manifold M, it fixes the Levi-Civita connection ${}^{LC}\nabla$. The holonomy group $Hol(g)$ is in general not identical to the structure group, but they are related by the notion of intrinsic torsion, which we introduce now.

Let ∇ be a connection in the tangent bundle over M and $X, Y \in \Gamma(TM)$ vector fields. The **torsion** of ∇ is defined as

$$T^{\nabla} := \nabla_X Y - \nabla_Y X - [X, Y] \in \Omega^2(M, TM). \tag{4.5}$$

We now identify the space of two-forms on M pointwise with the space of antisymmetric matrices, $\wedge^2 T_p^* M \cong \mathfrak{so}(d)$, via the map

$$\begin{aligned} \mathfrak{so}(d) &\to \wedge^2 T_p^* M \\ A &\mapsto \frac{1}{4} g_{AC} A^C{}_B e^{AB}. \end{aligned} \tag{4.6}$$

Denoting by $\mathfrak{g}$ the Lie algebra of a subgroup $G \subset SO(d)$, the Lie algebra $\mathfrak{so}(d)$ may be split as $\mathfrak{so}(d) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$ with respect to the metric induced by the Killing form. $\mathfrak{g}^{\perp}$ denotes the orthogonal complement of the Lie algebra $\mathfrak{g}$ in $\mathfrak{so}(d)$. In this situation, there exists a unique G-connection $\widetilde{\nabla}$, also referred to as **minimal G-connection**, that is metric-compatible and has holonomy $Hol(\widetilde{\nabla}) = G$ equal to the reduced structure group[6] . The torsion of $\widetilde{\nabla}$ is usually nonzero. It can be shown that the difference tensor

$$\mathcal{T} := \widetilde{\nabla} - {}^{LC}\nabla \tag{4.7}$$

is an element of $T_p^* M \otimes \mathfrak{g}^{\perp} \subset T_p^* M \otimes \wedge^2 T_p^* M$ at any point $p \in M$ [50]. $\mathcal{T}$ is called the **intrinsic torsion** of the G-structure manifold. If $\mathcal{T}$ vanishes, $\widetilde{\nabla}$ is equivalent

[6]The minimal G-connection is by definition equivalent to the canonical connection introduced in Chapter 3.

to the Levi-Civita connection, and the holonomy group $Hol(g)$ is contained in $Hol(\widetilde{\nabla}) = G$. Such a G-structure is also referred to as torsion-free or integrable[7]. In this sense, the intrinsic torsion measures the deviation of the holonomy group from being contained in the structure group. The intrinsic torsion can be used to classify G-structure manifolds. This is done by considering $T_p^*M \otimes \mathfrak{g}^\perp$ as a representation space of the structure group G and decomposing it into irreducible G-representations. This will be illustrated in Chapter 4.4 and 4.5 in the cases of $SU(3) \subset SO(6)$ and $G_2 \subset SO(7)$.

Alternative to their definition as a G-subbundle P of the frame bundle, G-structures can be determined by the existence of G-invariant, non-degenerate and globally defined tensors, a characterization that is more common in physics literature than the above one. The G-structure then arises as a restriction of the transition functions of the corresponding bundle by the requirement that they leave the defining objects invariant. The choice of invariant objects is usually not unique, implying that the same G-structure can be described by different sets of objects. If the structure group is reduced to $SO(d)$ and the manifold is spin, we may in particular use spinors as defining objects. Details about this definition and further examples can also be found in [68].

As already mentioned, there is a one-to-one correspondence between the existence of a metric and a reduction of the structure group to $O(d)$. The following other examples will be of interest: given an almost complex structure J (cf. Appendix A) on an even d-dimensional manifold M, the structure group reduces to $GL\left(\frac{d}{2}\right) \subset GL(d)$. If in addition the manifold admits a Hermitean metric satisfying the compatibility condition (A.13), the structure group further reduces to $U\left(\frac{d}{2}\right)$. As described in Appendix A, this implies in particular the existence of a fundamental $(1,1)$-form ω. It can be shown that it suffices to have two of the structures (ω, g, J) on a manifold to uniquely fix the third. In seven dimensions, the reduction to the structure group $G_2 \subset SO(7)$ is determined by the existence of a globally defined three-form φ. It can be shown that the differentials of the defining forms decompose into the same irreducible G-representations as the intrinsic torsion and that this decomposition may equally well be used to classify the G-structure.

As described, a compact manifold must admit a globally defined real Killing spinor to ensure the preservation of four-dimensional $\mathcal{N} = 1$ supersymmetry in

[7]From now on, we mean by "holonomy of a manifold" the holonomy of the Levi-Civita connection. If instead the holonomy of a torsionful connection is meant, this will be explicitly indicated.

Manifold M	$\dim(M)$	Structure group G
round spheres	n	$SO(n)$
Nearly-Kähler	6	$SU(3)$
Nearly parallel G_2	7	G_2
Sasaki-Einstein	$2m+1$	$SU(m)$
3-Sasakian	$4m+3$	$Sp(m)$

Table 2: Bär's list of Riemannian manifolds with real Killing spinors.

string compactifications. The existence of such a spinor determines a G-structure, but not all G-structure manifolds carry a real Killing spinor. Riemannian manifolds with real Killing spinors have been completely classified by Bär in [31], resulting in the list presented in Table 2. All these manifolds come equipped with a canonical three- and four-form determined by the spinor, all of them are Einstein, admit a non-integrable G-structure and a connection with nonvanishing torsion.

Bär's classification is based on the observation that the Riemannian cone over a real Killing spinor manifold has special (reduced) holonomy. Some examples are listed in Table 3. The following relations (cf. [28]) will be particularly interesting for our work:

- the metric cone over a Sasakian manifold is Kähler,
- the metric cone over a Sasaki-Einstein manifold is Calabi-Yau,
- the metric cone over a nearly-Kähler manifold has holonomy contained in G_2,
- the metric cone over a G_2-structure manifold has holonomy contained in $Spin(7)$.

In the compactification of higher-dimensional gauge theory, the existence of a parallel spinor is related to a condition on the gauge connection on the compact internal manifold: the connection is required to satisfy the higher-dimensional instanton equation. The exact relation between these conditions has been discussed

Real Killing spinor manifold M	$\dim(\mathcal{C}(M))$	Holonomy group of $\mathcal{C}(M)$
Sasaki-Einstein	d	$SU(d)$
3-Sasakian	d	$Sp(d)$
Nearly-Kähler	7	G_2
G_2	8	$Spin(7)$

Table 3: Real Killing spinor manifolds and the holonomy groups of their cones.

in [29]. On G-structure manifolds, covariant differentiation of the instanton equation implies the Yang-Mills equation with torsion. The torsion term vanishes on manifolds with real Killing spinor and in particular on manifolds with special holonomy.

4.3 Sasakian Manifolds

Let us take a closer look at some examples of G-structure manifolds that will be used in Parts II and III of this thesis. We start with Sasakian manifolds of dimension $2m+1$ with $1 \leq m \in \mathbb{N}$. They appear as compactification spaces in various higher-dimensional theories and, as described in [39], they provide a bridge between all other special geometries listed in Table 2. A detailed introduction to Sasakian geometry can be found in [39, 47, 52].

Sasakian manifolds are special types of contact manifolds. According to [53, 54], an **almost contact structure** (Φ, η, ξ) on an odd-dimensional Riemannian manifold (M, g) is characterized by a nowhere vanishing vector field $\xi \in \Gamma(TM)$ and a one-form $\eta \in \Omega^1(M)$, satisfying $\eta(\xi) = 1$, plus a $(1,1)$-tensor Φ such that $\Phi^2 = -\mathbb{1} + \xi \otimes \eta$. Such a structure is called **contact** if in addition the one-form satisfies

$$\eta \wedge (d\eta)^m \neq 0. \tag{4.8}$$

In this case, η is called **contact form**, and ξ is referred to as **Reeb vector field**. Contact structures are **normal** if for their Nijenhuis tensor associated to the tensor Φ,

$$N_\Phi(X,Y) = \Phi^2[X,Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] \qquad \forall\, X, Y \in \Gamma(TM), \tag{4.9}$$

the relation

$$N = -d\eta \otimes \xi \tag{4.10}$$

holds[8]. When the Riemannian metric g on an almost contact manifold (M, g) satisfies

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{4.11}$$

for any vector fields $X, Y \in \Gamma(TM)$, the structure is referred to as **almost contact metric**. It is called **contact metric** if in addition

$$d\eta = 2\omega, \tag{4.12}$$

with a two-form $\omega(X,Y) := g(X, \Phi Y)$, is satisfied.

A **Sasakian manifold** can now be defined as a manifold with normal contact metric structure.

Sasakian manifolds admit a reduction of the tangent bundle structure group from $SO(2m+1)$ to $U(m)$. In certain cases (provided that the manifold admits two Killing spinors $\epsilon, \tilde{\epsilon}$ [29]), the structure group can be further reduced to $SU(m)$. This structure allows – apart from the existence of the one-form $\eta \in \Omega^1(M)$ and two-form $\omega \in \Omega^2(M)$ – for the introduction of forms $P \in \Omega^3(M)$ and $Q \in \Omega^4(M)$ that satisfy the following relations:

$$P = \eta \wedge \omega, \qquad Q = \frac{1}{2}\omega \wedge \omega, \qquad \eta \lrcorner \omega = 0. \tag{4.13}$$

The contraction is defined as $\eta \lrcorner \omega = *(\eta \wedge *\omega)$ by use of the Hodge star operator on (M, g) (see for example [24]). All these forms are parallel with respect to the canonical connection that is specified below. In addition to (4.12), they satisfy the relations

$$d * \omega = 2m * \eta, \tag{4.14}$$

$$dP = 4Q, \tag{4.15}$$

$$d * Q = (2m-2) * P. \tag{4.16}$$

[8]This is equivalent to the complex structure J induced on the product manifold $\mathbb{R} \times M$ being integrable.

Condition (4.12) can be generalized. If the structure satisfies $d\eta = \alpha\omega$ for some real parameter $\alpha \in \mathbb{R}$, it is referred to as α-Sasakian. We will see below that the α-Sasakian structure can be transformed into a Sasakian structure by rescaling of basis elements.

It is useful to choose a local orthonormal basis $\{e^1, e^a\}$ of T^*M such that the parallel one- and two-form become

$$\eta = e^1, \qquad \omega = e^{23} + e^{45} + \ldots + e^{2m,2m+1}. \tag{4.17}$$

In order to distinguish the contact direction, we use a slightly different index convention than for the Lie groups in Chapter 2. We use small indices $a = (2, \ldots, 2m+1)$ to label directions on the Sasakian manifold excluding the contact direction. Indices that can be either 1 or a are labeled by Greek letters $\mu = (1, 2, \ldots, 2m+1)$. As described in Chapter 3, the torsion of the canonical connection is proportional to the three-form P. This connection has the following coefficients on a Sasakian manifold[9] [29]:

$$^{P}\Gamma^{b}_{\mu a} = {}^{LC}\Gamma^{b}_{\mu a} + \frac{1}{m} P_{\mu ab}, \tag{4.18}$$

$$^{P}\Gamma^{a}_{\mu 1} = -{}^{P}\Gamma^{1}_{\mu a} = {}^{LC}\Gamma^{a}_{\mu 1} + P_{\mu 1 a}. \tag{4.19}$$

The connection $^{P}\nabla$ is constructed such that the Killing spinors are parallel with respect to it, i. e. $^{P}\nabla\epsilon = 0$, and hence has holonomy $SU(m)$. It is compatible with the following family of metrics parametrized by a real constant h, all of which are Sasakian up to homothety:

$$g_h = e^1 e^1 + e^{2h}\delta_{ab} e^a e^b. \tag{4.20}$$

This can be seen by rescaling the metric with a real parameter γ,

$$g_{h,\gamma} = \gamma^2 (e^1 e^1 + e^{2h}\delta_{ab} e^a e^b), \tag{4.21}$$

and introducing new basis forms $\widetilde{e}^1 = \gamma e^1, \widetilde{e}^a = \gamma e^h e^a$, such that the rescaled metric takes the form

$$g_{h,\gamma} = \widetilde{e}^1 \widetilde{e}^1 + \delta_{ab} \widetilde{e}^a \widetilde{e}^b. \tag{4.22}$$

Recall that the original basis one-forms satisfy the Sasaki relation $de^1 = 2\omega$ (4.12). For still being Sasakian after rescaling, the new basis elements have to satisfy an

[9] Note that the identities in the second line hold only for $g_{\mu\nu} = \delta_{\mu\nu}$, i. e. when $\{e^1, e^a\}$ constitute a non-coordinate basis. In all other cases, the appearance of metric factors has to be taken into account when raising and lowering indices.

analogous condition. We find

$$d\widetilde{e}^1 = \frac{2}{\gamma e^{2h}}\widetilde{\omega} := \alpha(h,\gamma)\widetilde{\omega}. \tag{4.23}$$

The structure is therefore α-Sasakian for all $\alpha(h,\gamma)$ (hence also for all scaling factors γ) and Sasakian (i. e. $d\widetilde{e}^1 = 2\widetilde{\omega}$) for the special value $\alpha = 2$, or, equivalently, $\gamma = e^{-2h}$.

If the metric on a Sasakian manifold is proportional to the Ricci tensor, we have a **Sasaki-Einstein manifold**, whose structure group is necessarily $SU(m)$. These are the manifolds listed in Tables 2 and 3. Note that a Sasakian manifold with $SU(m)$ structure group need not necessarily be Einstein, as will be explained in further detail in Chapter 9. Our Sasakian manifold with metric (4.20) becomes Einstein for $h = 0$. The value

$$e^{2h} = \frac{2m}{m+1} \tag{4.24}$$

is special as well, as it makes the torsion of the canonical connection totally antisymmetric. We will restrict our consideration to the latter case in Chapter 9 and not study the Einstein case in detail in this thesis.

4.4 $SU(3)$-Structure Manifolds and Their Torsion Classes

Six-dimensional manifolds with structure group $SU(3)$ are particularly interesting in the context of ten-dimensional string theories. Torsionful $SU(3)$-structures appear, for example, in the construction of heterotic domain wall solutions in [35].

There are several ways to determine an $SU(3)$-structure. One is to fix a metric g and almost complex structure J, as described in Chapter 4.2, plus a complex $(3,0)$-form Ω that determines an orientation. These structures imply the existence of a $(1,1)$-form ω. Alternatively, it suffices to fix the pair (ω, Ω) to determine the $SU(3)$-structure. These forms uniquely fix g and J as described in [6]. We refer to (ω, Ω) as $SU(3)$-structure forms. It is also possible to fix the $SU(3)$-structure is by a globally defined spinor, which can be shown to uniquely determine the structure forms (ω, Ω). We will however not discuss the spinor approach in further detail here.

The spaces of two- and three-forms decompose into irreducible representations under the action of $SU(3)$. Being globally defined and by definition $SU(3)$-

invariant, the forms ω and Ω transform in one-dimensional $SU(3)$-subrepresentations. Decomposition of the spaces of differential forms into irreducible $SU(3)$-representations implies the following compatibility conditions:

$$\omega \wedge \omega \wedge \omega = -\frac{3}{4} i\Omega \wedge \bar{\Omega}, \tag{4.25}$$

$$\omega \wedge \Omega = 0. \tag{4.26}$$

Equation (4.25) defines a volume form on the six-dimensional manifold. The minus sign on the right-hand side has been chosen to match equations in [35], from where most of our conventions have been adopted[10]. In addition, the structure forms satisfy the following relations, where the notation $\Omega = \Omega^+ + i\Omega^-$ is used for the real and imaginary part of the structure three-form:

$$*\omega = \frac{1}{2}\omega \wedge \omega, \qquad *(\omega \wedge \omega) = 2\omega, \tag{4.27}$$

$$*\Omega^+ = \Omega^-, \qquad *\Omega^- = -\Omega^+. \tag{4.28}$$

The forms defining the $SU(3)$-structure are parallel with respect to the minimal G-connection. This is equivalent to

$$^{LC}\nabla\omega = -\mathcal{T}\omega, \qquad ^{LC}\nabla\Omega = -\mathcal{T}\Omega. \tag{4.29}$$

It can be shown [50] that on an $SU(3)$-structure manifold $^{LC}\nabla\omega$, $d\omega$ and $\mathcal{T}\omega$ decompose into the same irreducible $SU(3)$-representations. The intrinsic torsion of an $SU(3)$-structure manifold at a point $p \in M$ decomposes as follows:

$$\mathcal{T}^C_{AB} \in T^*_p M \otimes \mathfrak{su}(3)^\perp = \underbrace{(\mathbf{1} \oplus \mathbf{1})}_{\mathcal{W}_1} \oplus \underbrace{(\mathbf{8} \oplus \mathbf{8})}_{\mathcal{W}_2} \oplus \underbrace{(\mathbf{6} \oplus \bar{\mathbf{6}})}_{\mathcal{W}_3} \oplus \underbrace{2(\mathbf{3} \oplus \bar{\mathbf{3}})}_{\mathcal{W}_4, \mathcal{W}_5}, \tag{4.30}$$

where we label the representations by their real dimension and associate to each representation component a torsion class $\mathcal{W}_m$, $m = (1, \ldots, 5)$. $\mathcal{W}_1 = \mathcal{W}_1^+ + i\mathcal{W}_1^-$ is a complex scalar, $\mathcal{W}_2 = \mathcal{W}_2^+ + i\mathcal{W}_2^-$ is a complex, primitive[11] $(1,1)$-form, $\mathcal{W}_3$ is a real, primitive $((2,1)+(1,2))$-form, $\mathcal{W}_4$ is a real vector and $\mathcal{W}_5$ is a complex $(1,0)$-form. The structure forms (ω, Ω) are in general not closed, and we may express their differentials $d\omega \in \Omega^3(M)$ and $d\Omega \in \Omega^4(M)$ in terms of the same torsion classes as

$$d\omega = \frac{3}{4} i(\mathcal{W}_1\bar{\Omega} - \bar{\mathcal{W}}_1\Omega) + \mathcal{W}_4 \wedge \omega + \mathcal{W}_3, \tag{4.31}$$

$$d\Omega = -\mathcal{W}_1\omega \wedge \omega + \mathcal{W}_2 \wedge \omega + \bar{\mathcal{W}}_5 \wedge \Omega. \tag{4.32}$$

[10] We use the conventions of [35], but our structure constants are normalized such that they satisfy equations (2.23) to (2.25). This leads to a rescaling of $d\omega$, $d\Omega$, $\mathcal{W}_1$ and $\mathcal{W}_2$ compared to the equations in the reference.

[11] Primitivity means tracelessness with respect to ω: $(\mathcal{W}_2)_{AB}\omega^{AB} = 0$.

Manifold	Vanishing torsion classes
Complex	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
Symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
Half-flat	$\mathcal{W}_1^- = \mathcal{W}_2^- = \mathcal{W}_4 = \mathcal{W}_5 = 0$
restricted half-flat	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
Special Hermitean	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
Nearly-Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
Almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

Table 4: Some special $SU(3)$-structures, characterized by their torsion classes.

In the case of $d\omega = d\Omega = 0$, the manifold has $SU(3)$-holonomy, i. e. it is a Calabi-Yau manifold.

Table 4 contains an incomplete list of special types of $SU(3)$-structure manifolds and the corresponding torsion classes. We will be particularly interested in half-flat manifolds, which include the class of nearly-Kähler manifolds. On half-flat manifolds, the structure equations (4.25) and (4.26) imply

$$\omega \wedge d\omega = 0. \tag{4.33}$$

Nearly-Kähler manifolds have been of particular interest in the context of string compactifications, as they are an easily accessible generalization of Calabi-Yau manifolds, and a number of explicit examples of them are known. Their structure forms satisfy the following relations, in addition to the above presented structure equations:

$$d\omega \propto \Omega^-, \qquad d\Omega^+ \propto 2\omega \wedge \omega. \tag{4.34}$$

The Coset Space $SU(3)/(U(1)\times U(1))$

Let us take a closer look at the coset space $SU(3)/(U(1)\times U(1))$, which will be used in Chapter 7 for the study of explicit instanton solutions. This space comes with a restricted half-flat $SU(3)$-structure (cf. Table 4 and [60]) and nonvanishing torsion classes $\mathcal{W}_1^+, \mathcal{W}_2^+$. The most general $SU(3)$-invariant metric on $SU(3)/(U(1)\times U(1))$ takes the form

$$g_6 = R_1^2(e^1\otimes e^1 + e^2\otimes e^2) + R_2^2(e^3\otimes e^3 + e^4\otimes e^4) + R_3^2(e^5\otimes e^5 + e^6\otimes e^6) \tag{4.35}$$

with real constants $R_1, R_2, R_3 \in \mathbb{R}$. We find three $SU(3)$-invariant two-forms $\omega_1, \omega_2, \omega_3$ and two invariant three-forms ρ_1, ρ_2 that can be written in a local basis as

$$\omega_1 = e^{12}, \qquad \omega_2 = -e^{34}, \qquad \omega_3 = e^{56}, \tag{4.36}$$

$$\rho_1 = e^{136} - e^{145} + e^{235} + e^{246}, \qquad \rho_2 = e^{135} + e^{146} - e^{236} + e^{245}. \tag{4.37}$$

The $SU(3)$-structure forms (ω, Ω) can be written as linear combinations of these invariant forms:

$$\omega = R_1^2\omega_1 + R_2^2\omega_2 + R_3^2\omega_3, \tag{4.38}$$

$$\Omega = R_1R_2R_3(\rho_1 + i\rho_2). \tag{4.39}$$

The torsion classes are computed from the structure forms, using Cartan's equations (2.16) and (2.17) as well as the structure constants

$$\begin{aligned} f_{12}^7 &= \frac{1}{\sqrt{3}}, \\ f_{13}^6 &= -f_{14}^5 = f_{23}^5 = f_{24}^6 = f_{34}^7 = -f_{56}^7 = \frac{1}{2\sqrt{3}}, \\ f_{34}^8 &= f_{56}^8 = \frac{1}{2}. \end{aligned} \tag{4.40}$$

We find

$$d\omega = \frac{1}{2\sqrt{3}}(R_1^2 + R_2^2 + R_3^2)\rho_2, \tag{4.41}$$

$$d\Omega^+ = -\frac{2}{\sqrt{3}}R_1R_2R_3(\omega_1\wedge\omega_2 + \omega_1\wedge\omega_3 + \omega_2\wedge\omega_3), \tag{4.42}$$

$$d\Omega^- = 0, \tag{4.43}$$

and the torsion classes take the form

$$\mathcal{W}_1^+ = \frac{R_1^2 + R_2^2 + R_3^2}{3\sqrt{3}R_1R_2R_3}, \tag{4.44}$$

$$\mathcal{W}_2^+ = \frac{2}{3\sqrt{3}R_1R_2R_3}\big(R_1^2(2R_1^2 - R_2^2 - R_3^2)e^{12} - R_2^2(2R_2^2 - R_1^2 - R_3^2)e^{34} + R_3^2(2R_3^2 - R_1^2 - R_2^2)e^{12}\big)\,. \tag{4.45}$$

When all R's are equal, the second torsion class vanishes and the space becomes nearly-Kähler.

In addition, let us list the following identities, which are useful in explicit computations:

$$d(\omega \wedge \omega) = 0, \tag{4.46}$$

$$\omega \wedge \omega = 2(R_1^2R_2^2\omega_1 \wedge \omega_2 + R_2^2R_3^2\omega_2 \wedge \omega_3 + R_1^2R_3^2\omega_1 \wedge \omega_3), \tag{4.47}$$

$$\rho_1 \wedge \rho_2 = 4e^{123456}, \tag{4.48}$$

$$*\rho_1 = \rho_2. \tag{4.49}$$

4.5 G_2-Structure Manifolds

Closely related to six-dimensional $SU(3)$-structure manifolds are seven-dimensional manifolds with G_2-structure. As already mentioned, the cone over a nearly-Kähler manifold has holonomy contained in G_2, and the cylinder over an $SU(3)$-structure manifold comes with a torsionful G_2-structure. A G_2-structure on the manifold M is uniquely determined by the existence of a three-form P, allowing us to refer to such a manifold as (M, P). Being a subgroup of $SO(7)$, the structure group G_2 determines a metric g_7 and orientation on M.

An alternative characterization in terms of spinors is possible as well by lifting G_2 to a subgroup of $Spin(7)$. We will not describe the details here but note in this context that a seven-dimensional manifold admits a G_2-structure if and only if it is orientable and spin.

Together with the metric, the three-form uniquely determines a four-form $Q = *P$. We refer to the pair (P, Q) as structure forms of the G_2-structure, in analogy to the $SU(3)$-structure case. G_2-structure manifolds show strong similarities to almost Hermitean manifolds, more details of which can be found for instance in [69, 70]. Manifolds with G_2-structure are in particular Ricci-flat, which makes them interesting for compactifications of eleven-dimensional M-theory.

As before, we use the intrinsic torsion to classify G_2-structures. The intrinsic torsion at a point $p \in M$ decomposes into irreducible G_2-representations according to

$$\mathcal{T}_{AB}{}^{C} \in T_p^*M \otimes \mathfrak{g}_2^{\perp} = \underset{\tau_0}{\mathbf{1}} \oplus \underset{\tau_1}{\mathbf{7}} \oplus \underset{\tau_2}{\mathbf{14}} \oplus \underset{\tau_3}{\mathbf{27}}, \tag{4.50}$$

where we label the representation spaces by their dimension and associate a form $\tau_m, m = (0, \ldots, 3)$, to each of them. τ_0 is a scalar, τ_1 a one-form, τ_2 a two-form and τ_3 a three-form. The spaces of three- and four-forms on M decompose into the same irreducible subrepresentations, and the differentials of the structure forms (P, Q) may be written in terms of the torsion forms as

$$dP = \tau_0 Q + 3\tau_1 \wedge P + *\tau_3, \tag{4.51}$$

$$dQ = 4\tau_1 \wedge Q + *\tau_2. \tag{4.52}$$

Note that τ_1 appears in both decompositions, a fact that is proven in Theorem 2.23 of [70]. As there are four torsion classes, we find in total 16 classes of G_2-structures, which have been classified by Fernández and Gray in [71]. In particular, a G_2 structure is called torsion-free if both P and Q are closed:

$$dP = dQ = 0. \tag{4.53}$$

This is equivalent to the vanishing of all torsion classes. Some other interesting classes of G_2-structures according to [72] are listed in Table 5. In particular, a G_2-structure is called cocalibrated if the four-form is closed, $dQ = 0$, and nearly-parallel if in addition the three-form satisfies $dP \propto Q$. More details about G_2-structure torsion classes can also be found in [32, 72].

For the results presented in Chapter 7, it is important to understand the relation of the structure forms (P, Q) and the metric g_7 of a G_2-structure manifold. According to equation (A.9) in [62], the metric is determined by the three-form P via the relation

$$(g_7)_{AB} = -\frac{1}{144}\epsilon^{CDEFKMN} P_{ACD} P_{BEF} P_{KMN}, \tag{4.54}$$

where ϵ denotes the curved Levi-Civita symbol which takes values

$$\epsilon_{ABCDEFG} \in \{\pm\sqrt{|g_7|}\}. \tag{4.55}$$

Conventions for curved and flat Levi-Civita symbols are adopted from [67] and summarized in Appendix B. Solving equation (4.54) for $|g_7|$ and inserting the result

Manifold	Vanishing torsion classes
Nearly parallel	$\tau_1 = \tau_2 = \tau_3 = 0$
Almost parallel or coclosed	$\tau_0 = \tau_1 = \tau_3 = 0$
Balanced	$\tau_0 = \tau_1 = \tau_2 = 0$
Locally conformally parallel	$\tau_0 = \tau_2 = \tau_3 = 0$
Cocalibrated	$\tau_1 = \tau_2 = 0$
G_2-holonomy	$\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$

Table 5: Some classes of G_2-manifolds, characterized by their torsion classes.

back into the original formula allows for an explicit computation of the metric components. It seems that the three-form P is somehow more fundamental than the four-form Q, as it determines the metric g_7. However, the following arguments show that the four-form Q can be used equally well to determine the metric. In seven dimensions, a four-form fixes a three-form as follows:

$$S^{ABC} = \frac{1}{4!}\epsilon^{ABCDMNP}Q_{DMNP}. \tag{4.56}$$

As in equation (4.54), the inverse metric can be determined via this three-form as

$$(g_7)^{AB} = -\frac{1}{144}\epsilon_{CDEFKMN}S^{ACD}S^{BEF}S^{KMN}. \tag{4.57}$$

Replacing the components of S by the components of Q using equation (4.56) yields

$$(g_7)^{AB} = -\left(\frac{1}{4!}\right)^3 \epsilon^{AMNX_1\cdots X_4}\epsilon^{BPQY_1\cdots Y_4}Q_{X_1\cdots X_4}Q_{Y_1\cdots Y_4}Q_{MNPQ}. \tag{4.58}$$

Expressing the curved Levi-Civita symbols by flat ones and moving the determinant to the left-hand side allows to determine $|g_7|$. Inserting the result back yields an explicit expression for $(g_7)^{AB}$ in terms of the components of Q.

5 Yang-Mills Action and Self-Duality in Higher Dimensions

In this section, we introduce the Yang-Mills equation and the notion of instantons in dimensions higher than four. We list some properties of the instanton equation, including its relation to the geometric structures introduced in the previous chapter.

Let $E = P \times_\rho V$ be a vector bundle associated to the principal bundle $P(M, G)$ with respect to some representation ρ of G. Let $\mathcal{A}$ be a local connection form on E with curvature $\mathcal{F}$. We introduce the **Yang-Mills action** as

$$S_{YM} = \int_M \mathrm{tr}\ (\mathcal{F} \wedge *\mathcal{F}) = \frac{1}{2} \int_M \mathrm{tr}\ (\mathcal{F}_{AB} \mathcal{F}^{AB}) Vol(M), \tag{5.1}$$

where the trace is taken over the representation indices of the $\mathfrak{g}$-generators in the representation ρ. We denote by $Vol(M) = \sqrt{|g|} e^1 \wedge \cdots \wedge e^d$ the volume form of the base space (M, g) and by $*$ the Hodge star operator with respect to g. It can be shown that this action is invariant under gauge transformations.

The Yang-Mills action gives rise to the following definition: A connection form $\mathcal{A}$ is called **Yang-Mills connection** if its curvature $\mathcal{F}$ satisfies the **Yang-Mills equation**

$$\mathcal{D} * \mathcal{F} = 0. \tag{5.2}$$

The Yang-Mills equation is the equation of motion for the action (5.1) and takes the following form in components:

$$\mathcal{D}_A \mathcal{F}^{AB} = \partial_A \mathcal{F}^{AB} + [\mathcal{A}_A, \mathcal{F}^{AB}] = 0, \tag{5.3}$$

where we use capital indices to label directions on M. As this is a second-order differential equation, the construction of analytic solutions in explicit examples is not always possible. We can, however, find gauge connections that satisfy a first-order differential equation and minimize the Yang-Mills action functional. The solutions to the first-order equation solve the second-order one, but it is usually not possible to find all second-order solutions by solving the first-order equation.

To understand the idea, let us restrict the discussion to $M = \mathbb{R}^4$ with Euclidean metric $g_{AB} = \delta_{AB}$ for the moment. The concept can be generalized to higher dimensions and more complicated geometries after introducing additional structure. In four dimensions, the Hodge star operator maps two-forms to two-forms, which allows for the following definition:

A two-form $\omega \in \Omega^2(M)$ is called **self-dual** if it satisfies the condition $*\omega = \omega$. It is called **anti-self-dual** if it satisfies $*\omega = -\omega$. A local connection form $\mathcal{A} \in \Omega^1(U, \mathfrak{g})$ on some small open subset $U \subset M$ is called **(anti-)self-dual** if its curvature $\mathcal{F} \in \Omega^2(U, \mathfrak{g})$ is (anti-)self-dual.

The space $\Omega^2(M)$ of two-forms on a four-dimensional manifold M splits into a direct sum of self-dual $\Omega^2_+(M)$ and anti-self dual $\Omega^2_-(M)$ forms according to

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M). \tag{5.4}$$

In this context, the Hodge star operator can be understood as an operator with eigenvalues ± 1. Self-dual and anti-self-dual two-forms are eigenforms of the $+1$ and -1 eigenspaces, respectively. The equation

$$*\mathcal{F} = \pm \mathcal{F} \tag{5.5}$$

is called **(anti-)self-duality** or **instanton** equation. Finite-action solutions of equation (5.5) are referred to as **instantons**.

Covariant differentiation of $*\mathcal{F} = \pm\mathcal{F}$ and use of the Bianchi identity (3.15) lead to the Yang-Mills equation, implying that every (anti-)self-dual connection is in particular a Yang-Mills connection.

The Yang-Mills action has a lower bound, also known as BPS bound, which is saturated if the curvature $\mathcal{F}$ is (anti-)self-dual. To see this, we rewrite the action as

$$\begin{aligned} S_{YM} &= \frac{1}{2}\int_M d^4x \operatorname{tr}\left(\mathcal{F}_{AB}\mathcal{F}^{AB}\right) \\ &= \frac{1}{4}\int_M d^4x \left(\operatorname{tr}\left((\mathcal{F}_{AB} \mp *\mathcal{F}_{AB})(\mathcal{F}^{AB} \mp *\mathcal{F}^{AB})\right) \pm 2\operatorname{tr}\left(\mathcal{F}_{AB} * \mathcal{F}^{AB}\right)\right) \\ &= \frac{1}{4}\int_M d^4x \left(\operatorname{tr}\left(\mathcal{F}_{AB} \mp *\mathcal{F}^{AB}\right)^2 \pm 2\operatorname{tr}\left(\mathcal{F}_{AB} * \mathcal{F}^{AB}\right)\right) \\ &\geq \pm\frac{1}{2}\int_M d^4x \operatorname{tr}\left(\mathcal{F}_{AB} * \mathcal{F}^{AB}\right), \end{aligned} \tag{5.6}$$

using $*\mathcal{F}_{AB} = \frac{1}{2}\epsilon_{ABCD}\mathcal{F}^{CD}$ and $g_{AB} = \delta_{AB}$. The inequality turns into equality when the curvature satisfies the instanton equation (5.5). This implies that instantons minimize the Yang-Mills action, in agreement with the above observation that they are in particular Yang-Mills connections. A solution for the four-dimensional instanton equation with structure group $G = SU(2)$ can be found, for example, in [56].

The idea of replacing the second-order Yang-Mills equation by a first-order condition can be generalized to higher dimensions after introducing additional structure. Given a G-structure on the d-dimensional Riemannian manifold (M, g) and assuming that G is simple, it is possible to construct a globally defined, G-invariant four-form Q by taking the inverse of the Killing form and using the fact that the space of two-forms $\Omega^2(M)$ is pointwise isomorphic to $\mathfrak{so}(d)$ (cf. [29]). It turns out that this four-form vanishes if $G = SO(d)$ but is nonzero if $G \subset SO(d)$ is a proper subgroup. Given such a Q, we may construct an operator

$$\begin{aligned} *(*Q \wedge \cdot) : \Omega^2(M) &\to \Omega^2(M) \\ \eta &\mapsto *(*Q \wedge \eta) \end{aligned} \tag{5.7}$$

that commutes with the action of G. It follows that the irreducible subrepresentations of G in the space of two-forms are eigenspaces of this operator. The eigenvalue of the adjoint representation of G can be normalized to -1, and the other eigenvalues are determined case-by-case. They have been listed for the cases most interesting for us in [57]. **Higher-dimensional instantons** are then defined to be two-forms that transform in the -1 eigenspace of the operator[12] $*(*Q \wedge \cdot)$. The higher-dimensional instanton equation takes the form

$$*\mathcal{F} = - * Q \wedge \mathcal{F}. \tag{5.8}$$

Covariant differentiation of this equation implies the Yang-Mills equation with torsion,

$$d * \mathcal{F} + [\mathcal{A}, *\mathcal{F}] + *\mathcal{H} \wedge \mathcal{F} = 0, \tag{5.9}$$

where $*\mathcal{H} := d{*}Q$. The torsion term vanishes if $d{*}Q \wedge \mathcal{F} = 0$, which is in particular the case if the underlying manifold M has special holonomy. The generalized instanton equation implies the torsionful Yang-Mills equation and extremizes the

[12] More generally, instantons could be defined to be two-forms that belong to any of the eigenspaces of this operator. Then they satisfy the equation $*\mathcal{F} = \nu * Q \wedge \mathcal{F}$ for some real constant $\nu \in \mathbb{R}$ (cf. [29]). We will not use this more general definition here.

following action, which includes the familiar Yang-Mills plus a Chern-Simons term:

$$S = \int_M \text{tr}\ \left(\mathcal{F} \wedge *\mathcal{F} + (-1)^{d-3} * Q \wedge \mathcal{F} \wedge \mathcal{F}\right). \tag{5.10}$$

In this equation, d is the dimension of the manifold. In accordance with the four-dimensional case, we require the higher-dimensional instantons to have finite action as well, i. e. $S < \infty$ with S given by (5.10). It has been pointed out in [29, 57] that the instanton equation (5.8) is equivalent to requiring the two-form $\mathcal{F}$ to be contained in the Lie algebra $\mathfrak{g}$ of the structure group G after identifying $\wedge^2 T_p^* M \cong \mathfrak{so}(d)$. Furthermore, (5.8) implies the BPS condition $\gamma(\mathcal{F}) \cdot \varepsilon = 0$ (1.5) that arises in heterotic supergravity. In particular, imposing the higher-dimensional instanton equation requires the existence of a G-structure on M.

Let us mention some additional properties of the higher-dimensional instanton equation that illustrate its meaning as an eigenvalue equation. On almost complex manifolds, there is a special type of natural first-order BPS equations that has been introduced in [58, 59]. These equations are known as Hermitean Yang-Mills (HYM) or Donaldson-Uhlenbeck-Yau (DUY) equations. They are conditions for unbroken supersymmetry in heterotic string compactifications on compact Kähler manifolds, generalize the four-dimensional self-duality equations and imply the Yang-Mills equation. Let (M, g) be an even-dimensional manifold that carries and almost complex structure J, and $\mathcal{E}(M, V)$ a complex vector bundle endowed with a connection $\mathcal{A}$ with curvature $\mathcal{F}$. Then the HYM equations take the form

$$\mathcal{F}^{2,0} = \mathcal{F}^{0,2} = 0, \qquad\qquad \omega \lrcorner \mathcal{F} = 0. \tag{5.11}$$

$\mathcal{F}$ splits according to equation (A.7) into $\mathcal{F} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2}$. The equations (5.11) coincide with the anti-self-duality equation $*\mathcal{F} = -\mathcal{F}$ in four dimensions.

The generalized anti-self-duality equation on a six-dimensional $SU(3)$-structure manifold takes the form $*\mathcal{F} = -\omega \wedge \mathcal{F}$, where ω is the $SU(3)$-structure two-form. In this dimension, the space of two-forms is 15-dimensional. According to [24, 57], it decomposes into three eigenspaces $\Omega^2_\lambda(M)$ with eigenvalues $\lambda = \{-1, 1, 2\}$ of respective dimensions 8, 6 and 1. The -1 eigenspace contains the component $\mathcal{F}^{(1,1)}$ of holomorphicity degree $(1, 1)$, orthogonal to the $SU(3)$-structure form ω. The 1 eigenspace contains the components $\mathcal{F}^{(2,0)}$ and $\mathcal{F}^{(0,2)}$, and the two-forms in the 2 eigenspace are proportional to ω. In this case, the generalized instanton equation is equivalent to the HYM equations.

Equation (5.8) is invariant under conformal transformations [29, 55]. In particular, it takes the same form on the cone $\mathcal{C}(M)$ and on the cylinder $\mathcal{Z}(M)$ over a manifold M. The Hodge star operators on $\mathcal{C}$ and $\mathcal{Z}$, acting on a p-form, are related as

$$*_{\mathcal{C}} = e^{(d-2p)\tau} *_{\mathcal{Z}}, \tag{5.12}$$

with d denoting the dimension of $\mathcal{C}$ and $\mathcal{Z}$. This implies that the left-hand side of the instanton equation transforms as $*_{\mathcal{C}}\mathcal{F} = f^{d-4} *_{\mathcal{Z}}\mathcal{F}$. As the curvature is metric-independent, the components of $\mathcal{F}$ are equal on both manifolds. Both the cone and the cylinder admit a four-form Q. Conformal invariance then enforces the four-forms on $\mathcal{C}$ and $\mathcal{Z}$ to transform as $Q_{\mathcal{C}} = e^{4\tau} Q_{\mathcal{Z}}$ and imply that the instanton equations on both manifolds differ by a global conformal factor f^{d-4}:

$$*_{\mathcal{C}}\mathcal{F} + *_{\mathcal{C}}Q_{\mathcal{C}} \wedge \mathcal{F} = f^{d-4}(*_{\mathcal{Z}}\mathcal{F} + *_{\mathcal{Z}}Q_{\mathcal{Z}} \wedge \mathcal{F}) = 0. \tag{5.13}$$

Note that the Yang-Mills equation is not conformally invariant, a fact that will become important in Chapter 8.

Part II

Instantons on Coset Spaces

6 Instanton Equation on the Cylinder over a Coset Space

After introducing the most important facts about compact manifolds with G-structure and higher-dimensional Yang-Mills theory, let us now turn to the construction of explicit solutions of the higher-dimensional instanton equation. We start by studying instantons on the cylinder over a general reductive homogeneous space G/H and specialize to an explicit example later on.

Let G/H be an n-dimensional reductive homogeneous space and denote by $d = n+1$ the dimension of the product space $M = \mathbb{R} \times G/H$. For the construction of explicit instanton solutions, we first have to determine a four-form Q in this general setup. When explicit examples of G-structure coset spaces are considered later on, the four-form Q on M will be explicitly given by a combination of the structure forms. We start by expanding the instanton equation

$$*\mathcal{F} = - * Q \wedge \mathcal{F}. \tag{6.1}$$

in components. Before doing that, it is useful to apply the Hodge star operator once again. This yields

$$\mathcal{F} = - * (*Q \wedge \mathcal{F}), \tag{6.2}$$

using the fact that

$$* * \eta = (-1)^{r(d-r)} \eta \tag{6.3}$$

holds for any r-form η on a d-dimensional Riemannian manifold with positive signature, and in particular $** = id$ for differential forms of even degree. We find

$$*Q = \frac{1}{4!(d-4)!} Q^{ABCD} \epsilon_{ABCDM_5\cdots M_d} e^{M_5\cdots M_d} \tag{6.4}$$

and

$$*Q \wedge \mathcal{F} = \frac{1}{2\cdot 4!(d-4)!} Q^{ABCD} \epsilon_{ABCDM_5\cdots M_d} \mathcal{F}_{PQ} e^{M_5\cdots M_d PQ}, \tag{6.5}$$

where capital indices[13] $A = (0, \ldots, n)$ are used to label all directions on $\mathbb{R} \times G/H$ and $\epsilon_{ABCDM_5\cdots M_d}$ is the curved Levi-Civita symbol according to equation (B.3). Applying the Hodge star operator again, we find

$$*(*Q \wedge \mathcal{F}) = \frac{1}{2\cdot 2!4!(d-4)!} Q_{ABCD} \epsilon^{ABCDM_5\cdots M_d} \mathcal{F}^{PQ} \epsilon_{M_5\cdots M_d PQRS} e^{RS}. \tag{6.6}$$

Contraction of the Levi-Civita-symbols according to equation (B.7) and renaming of indices yield

$$\begin{aligned} *(*Q \wedge \mathcal{F}) &= \frac{1}{2\cdot 2!} Q_{ABCD} \mathcal{F}^{PQ} \delta^{ABCD}_{[PQRS]} e^{RS} \\ &= \frac{1}{2\cdot 2!} Q_{AB}{}^{CD} \mathcal{F}_{CD} e^{CD}. \end{aligned} \tag{6.7}$$

A comparison to the components of the left-hand side of equation (6.2) leads to

$$\mathcal{F}_{AB} = -\frac{1}{2} Q_{AB}{}^{CD} \mathcal{F}_{CD} \qquad \Leftrightarrow \qquad \mathcal{F}_{AB} = -\frac{1}{2} g^{CE} g^{DF} Q_{ABEF} \mathcal{F}_{CD}. \tag{6.8}$$

This form of the generalized instanton equation in components has first been introduced in [13]. Decomposition of the free indices A, B into directions on $\mathbb{R}$ and on G/H leads to two equations, a first-order differential equation and an algebraic condition, which will also be referred to as quiver relation. This relation allows to determine Q on a general space G/H and hence solve the first-order equation. On special cosets where Q is explicitly known, the quiver relation constitutes additional constraints that restrict possible instanton solutions.

With indices in the original position, equation (6.8) holds both on the cone and on the cylinder, using the respective four-form $Q_{\mathcal{Z}}$ or $Q_{\mathcal{C}}$. Because of the conformal invariance of the instanton equation, we may specialize to the cylinder with metric $g = d\tau^2 + \delta_{ab} e^{ab}$ and write out the instanton conditions explicitly.

[13] We use capital indices for directions on the product manifold $\mathbb{R} \times G/H$ and small indices $a = (1, \ldots, n)$ for directions on G/H. This is not to be confused with indices $\widetilde{a} = (1, \ldots, \dim G)$ used in Chapter 2 to label directions on the group manifold G.

With the aim to keep the solution as general as possible, we have to construct the four-form $Q_{\mathcal{Z}}$ without introducing any additional structure. Being a quotient of Lie groups, the manifold G/H is equipped with structure constants which are by definition antisymmetric in the lower two indices. Lowering the upper index of a structure constant with the Killing metric (2.19) leads to a totally antisymmetric object: $f^e_{ab}g_{ec} = f_{abc} = f_{[abc]}$ (see Appendix C.1 for details). Note that this property is lost when coset spaces with more general metric are considered. In our case, we can use the structure constants to define a three-form on $\mathcal{Z}(G/H)$ as

$$f := \frac{1}{3!} f_{abc} e^{abc}, \tag{6.9}$$

where $\{e^a\}$ is a basis of non-holonomic one-forms on $T^*(G/H)$ as in Chapter 2. Once a three-form is fixed, a four-form Q on $\mathcal{Z}(G/H)$ can be constructed as follows, with functions $\beta_1(\tau), \beta_2(\tau)$ that are to be determined:

$$Q := \beta_1(\tau) d\tau \wedge f + \beta_2(\tau) df. \tag{6.10}$$

Using the Maurer-Cartan equation (2.16)

$$de^a = -\frac{1}{2} f^a_{bc} e^{bc} - f^a_{ic} e^{ic} = -\left(\frac{1}{2} f^a_{bc} + f^a_{ic} e^i_b\right) e^{bc}, \tag{6.11}$$

the differential of f becomes

$$df = -\left(\frac{1}{4} f_{abc} f^a_{de} + \frac{1}{2} f_{abc} f^a_{ie} e^i_d\right) e^{debc}. \tag{6.12}$$

The last summand $f_{a[bc} f^a_{|i|e} e^i_{d]}$ vanishes by use of the Jacobi identity (C.2), and Q takes the form

$$Q = \frac{1}{3!} \beta_1 f_{abc} d\tau \wedge e^{abc} - \frac{1}{4} \beta_2 f_{abc} f^a_{de} e^{debc}. \tag{6.13}$$

Raising and lowering indices with the cylinder metric does not lead to any additional factors. Hence equation (6.8) takes the following form on $\mathcal{Z}(G/H)$:

$$\mathcal{F}_{AB} = -\frac{1}{2} Q_{ABCD} \mathcal{F}_{CD}. \tag{6.14}$$

Separating the indices $A = (0, a)$ yields a first-order equation and an algebraic condition. With the explicit components of Q, we obtain

$$\mathcal{F}_{0a} = -\frac{1}{2} Q_{0acd} \mathcal{F}_{cd} = -\frac{\beta_1}{2} f_{acd} \mathcal{F}_{cd}, \tag{6.15}$$

$$\begin{aligned} \mathcal{F}_{ab} &= -\left(Q_{ab0d} \mathcal{F}_{0d} + \frac{1}{2} Q_{abcd} \mathcal{F}_{cd}\right) \\ &= \frac{1}{2} Q_{0abe} Q_{0cde} \mathcal{F}_{cd} - \frac{1}{2} Q_{abcd} \mathcal{F}_{cd} \\ &= \left(\frac{1}{2} \beta_1^2 f_{abe} f_{cde} + 3\beta_2 f_{e[ab} f^e_{cd]}\right) \mathcal{F}_{cd}. \end{aligned} \tag{6.16}$$

6.1 Gauge Connection with One Scalar Function

To determine the functions $\beta_1(\tau), \beta_2(\tau)$, we need to further specify the components of the curvature form $\mathcal{F}$. We start with the simplest case, in which the gauge connection (3.31) is parametrized by on one real scalar function[14], $\mathcal{A} = e^i I_i + \phi(\tau) e^a I_a$. In this case, the curvature takes the form

$$\mathcal{F}_{0a} = \dot{\phi} I_a, \tag{6.17}$$

$$\mathcal{F}_{ab} = (\phi^2 - 1) f^i_{ab} I_i + (\phi^2 - \phi) f^f_{ab} I_f, \tag{6.18}$$

where $\dot{\phi} = \frac{\partial \phi(\tau)}{\partial \tau}$, and we can determine the functions $\beta_1(\tau), \beta_2(\tau)$ explicitly. Omitting the explicit τ-dependences, equation (6.16) turns into

$$\begin{aligned}(\phi^2 - 1) f^i_{ab} I_i &+ (\phi^2 - \phi) f^f_{ab} I_f \\ &= \left(\frac{1}{2}\beta_1^2 f_{abe} f_{cde} + 3\beta_2 f_{e[ab} f^e_{cd]}\right)\left((\phi^2 - 1) f^i_{cd} I_i + (\phi^2 - \phi) f^f_{cd} I_f\right).\end{aligned} \tag{6.19}$$

In order to determine β_1 and β_2, we need to assume in addition that the structure constants with indices in the original position are cyclic and satisfy

$$f^c_{ad} f^d_{cb} = \alpha \delta_{ab}, \tag{6.20}$$

$$f^i_{ad} f^d_{ib} = \frac{1}{2}(1 - \alpha)\delta_{ab}, \tag{6.21}$$

$$f^c_{ad} f^d_{ci} = \delta_{ai} = 0, \tag{6.22}$$

with some real parameter α specific to the chosen coset (cf. Chapter 2). Detailed derivations of the identities used in the following steps can be found in Appendix C.1. Note that the summand proportional to $f^e_{ab} f^e_{cd} f^i_{cd}$ in equation (6.19) vanishes due to equation (6.22) and that the antisymmetrized combination of structure constants $f_{e[ab} f^e_{cd]}$ satisfies

$$f_{e[ab} f^e_{cd]} = f_{e[ab} f^e_{c]d} = \frac{1}{3}\left(2 f_{ea[b} f^e_{c]d} + f_{e[bc]} f^e_{ad}\right). \tag{6.23}$$

Together with equations (C.7) and (C.8), we find

$$f_{e[ab} f^e_{cd]} f^i_{cd} = \frac{1}{3}\left(2 f_{ec[a} f^e_{b]d} + f_{e[ab]} f^e_{cd}\right) f^i_{cd} = -\frac{\alpha}{3} f^i_{ab}. \tag{6.24}$$

The combination $f^e_{ab} f^e_{cd} f^f_{cd}$ can be easily simplified using equation (6.20):

$$f^e_{ab} f^e_{cd} f^f_{cd} = \alpha f^f_{ab}. \tag{6.25}$$

[14] We omit the tensor product in the connection from now on.

For the last summand in equation (6.19), we use equation (C.16) and find

$$f_{e[ab}f^{e}_{cd]}f^{f}_{cd} = \frac{1}{6}(1-\alpha)f^{f}_{ab}. \tag{6.26}$$

With equations (6.23) to (6.26), equation (6.19) simplifies to

$$\begin{aligned}(\phi^2-1)f^{i}_{ab}I_i + (\phi^2-\phi)f^{f}_{ab}I_f \\ = -\alpha\beta_2(\phi^2-1)f^{i}_{ab}I_i + \left(\frac{\alpha\beta_1^2}{2}+\frac{(1-\alpha)\beta_2}{2}\right)(\phi^2-\phi)f^{f}_{cd}I_f.\end{aligned} \tag{6.27}$$

As the Lie group generators I_i and I_f are linearly independent, this yields two conditions:

$$\alpha\beta_2 = -1 \qquad \text{for } I_i, \tag{6.28}$$

$$\frac{\alpha\beta_1^2}{2}+\frac{(1-\alpha)\beta_2}{2} = 1 \qquad \text{for } I_f. \tag{6.29}$$

We find that on the cylinder, β_1 and β_2 are in fact τ-independent coefficients:

$$\beta_2(\tau) = \beta_2 = -\frac{1}{\alpha}, \tag{6.30}$$

$$\beta_1(\tau) = \beta_1 = \pm\frac{\sqrt{1+\alpha}}{\alpha}. \tag{6.31}$$

With these results, Q can be written down explicitly:

$$Q_{\mathcal{Z}} = \pm\frac{1}{3!}\frac{\sqrt{1+\alpha}}{\alpha}f_{abc}d\tau\wedge e^{abc} + \frac{1}{4\alpha}f_{e[ab}f^{e}_{cd]}e^{abcd}. \tag{6.32}$$

Inserting β_1 into the flow equation (6.15) and using equations (6.17) and (6.18) for $\mathcal{F}$ leads to

$$\mathcal{F}_{0a} = \mp\frac{\sqrt{1+\alpha}}{2\alpha}f^{c}_{ab}\mathcal{F}_{cd} \qquad \Leftrightarrow \qquad \dot{\phi} = \mp\frac{\sqrt{1+\alpha}}{2}(\phi^2-\phi). \tag{6.33}$$

For both overall signs, equation (6.33) yields kink-type solutions that interpolate between $\phi = 0$ and $\phi = 1$ for $\tau \to \pm\infty$:

$$\phi(\tau) = -\frac{1}{2}\tanh\left(\mp\frac{\sqrt{1+\alpha}}{4}(\tau-\tau_0)\right)+\frac{1}{2}. \tag{6.34}$$

These solutions are plotted in Figure 1 for $\tau_0 = 0$ and $\alpha = \frac{1}{3}$, which is the right value for the coset space $SU(3)/(U(1)\times U(1))$ with structure constants as in equation (4.40). Solutions of this type in similar setups have been found in earlier works, for example in [23, 24, 26, 28].

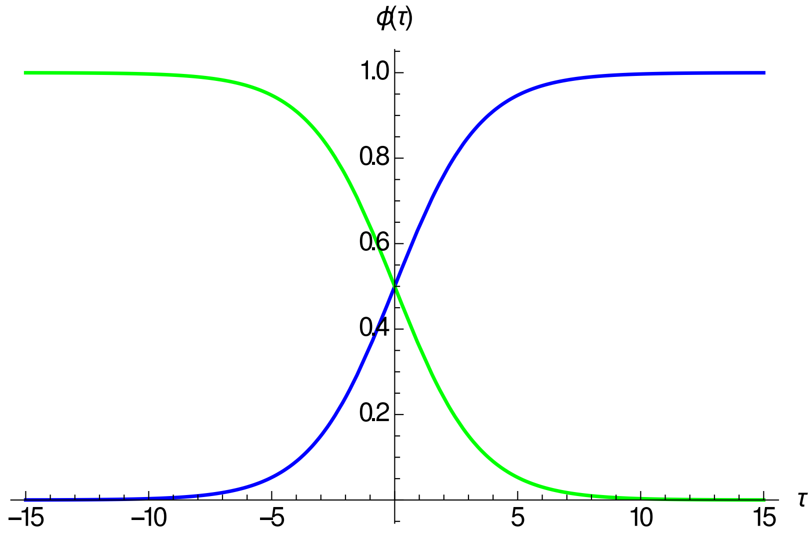

Figure 1: Plot of solution (6.34) for positive (green) and negative (blue) sign of the argument, with $\alpha = \frac{1}{3}$ and $\tau_0 = 0$.

6.2 Finiteness of the Yang-Mills Action

As already mentioned in Chapter 5, instantons are defined to be finite-action solutions of the Yang-Mills equation. Some comments about finiteness of the action are in order. Using the ansatz (6.17), (6.18), the action (5.10) can be written in the form

$$S = \int (T - V) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{\phi}^2 - V \right) d\tau, \tag{6.35}$$

where T denotes the kinetic term and V the potential, determined by explicitly writing out the expression $\mathcal{F} \wedge *\mathcal{F} + *Q \wedge \mathcal{F} \wedge \mathcal{F}$ as in Appendix C.3. This action is finite if the solution (6.34) interpolates between zero-potential critical points. Writing the corresponding Yang-Mills equation of motion as $\ddot{\phi} = -\partial_\phi V$ allows for the determination of the potential up to a constant term. The equations of motion, arising from variation of the action, do not change after an arbitrary constant is added to the potential V. In the cases considered here, one can always find a constant such that a given instanton solution has finite action, and the finiteness requirement can be used to determine this constant.

On the other hand, equation (5.10) together with the explicit forms of Q and $\mathcal{F}$ (resp. $\mathcal{A}$) can be used to determine the constant term in V. A computation of this type is explicitly presented for the cylinder over a Sasakian manifold in Appendix C.3, where the equations of motion (9.22) can be written in terms of a gradient system.

6.3 General G-Invariant Gauge Connection

Let us generalize the ansatz for the gauge connection and assume that it takes the form $\mathcal{A} = e^i I_i + e^a X_a$ subject to the G-invariance condition (3.33). This ansatz allows $\mathcal{A}$ to have more degrees of freedom and therefore leads to more complicated equations. The components of the curvature are given by

$$\mathcal{F}_{0a} = \dot{X}_a, \tag{6.36}$$

$$\mathcal{F}_{bc} = -(f^i_{bc} I_i + f^a_{bc} X_a - [X_b, X_c]). \tag{6.37}$$

Written out in components, the commutator takes the form

$$[X_a, X_b] = X^c_a X^d_b (f^i_{cd} I_i + f^e_{cd} I_e). \tag{6.38}$$

Inserting this ansatz for $\mathcal{F}$ into the first-order equation (6.15) yields

$$\dot{X}^b_a I_b = \frac{\beta_1}{2}\left(\alpha X^b_a I_b - X^m_c X^n_d (f_{acd} f^i_{mn} I_i + f_{acd} f^b_{mn} I_b)\right). \tag{6.39}$$

Comparing coefficients leads to the following two conditions:

$$\dot{X}^b_a = \frac{\beta_1}{2}\left(\alpha X^b_a - X^m_c X^n_d f_{acd} f^b_{mn}\right), \tag{6.40}$$

$$0 = \beta_1 X^m_c X^n_d f_{acd} f^i_{mn}. \tag{6.41}$$

The algebraic instanton condition (6.16) turns into the following equation by use of the identities (6.20) to (6.22) as well as (6.24) to (6.26):

$$f^i_{ab} I_i + f^f_{ab} X^e_f I_e - [X_a, X_b] = -\alpha\beta_2 f^i_{ab} I_i + \left(\frac{\beta_1^2 \alpha}{2} + \frac{\beta_2}{2}(1-\alpha)\right) f^m_{ab} X^e_m I_e - \left(\frac{\beta_1^2}{2} f_{abe} f_{cde} + 3\beta_2 f_{e[ab} f^e_{cd]}\right)[X_c, X_d]. \tag{6.42}$$

With the explicit expression for the commutator (6.38), this yields the following two conditions, one for each independent coefficient:

$$f^i_{ab} - X^c_a X^d_b f^i_{cd} = -\alpha\beta_2 f^i_{ab} - X^m_c X^n_d \left(\frac{\beta_1^2}{2} f_{abe} f_{cde} f^i_{mn} + 3\beta_2 f_{e[ab} f^e_{cd]} f^i_{mn}\right), \tag{6.43}$$

$$f^d_{ab} X^e_d - X^c_a X^d_b f^e_{cd} = \left(\frac{\alpha\beta_1^2}{2} + \frac{\beta_2(1-\alpha)}{2}\right) f^m_{ab} X^e_m - X^m_c X^n_d \left(\frac{\beta_1^2}{2} f_{abp} f_{cdp} f^e_{mn} + 3!\beta_2 f_{p[ab} f^p_{cd]} f^e_{mn}\right). \tag{6.44}$$

The first expression simplifies by use of equation (6.41) and takes the form

$$f^i_{ab} - X^c_a X^d_b f^i_{cd} = -\alpha\beta_2 f^i_{ab} - 3\beta_2 X^m_c X^n_d f_{e[ab} f^e_{cd]} f^i_{mn}. \tag{6.45}$$

As the canonical connection $\mathcal{A} = e^i I_i$ is an instanton, the relations (6.40), (6.41), (6.44) and (6.45) must be identically satisfied for these components of $\mathcal{A}$. This is the case if

$$\beta_2 = -\frac{1}{\alpha}, \tag{6.46}$$

as in the previous chapter. The algebraic relations therefore take the form

$$X_a^c X_b^d f_{cd}^i = -\frac{3}{\alpha} X_c^m X_d^n f_{e[ab} f_{cd]}^e f_{mn}^i, \tag{6.47}$$

$$\begin{aligned} X_a^c X_b^d f_{cd}^e = &-\left(\frac{\alpha^2 \beta_1^2 - (1+\alpha)}{2\alpha}\right) f_{ab}^d X_d^e \\ &+ \frac{\beta_1^2}{2} X_c^m X_d^n f_{abp} f_{cdp} f_{mn}^e - \frac{3!}{\alpha} X_c^m X_d^n f_{p[ab} f_{cd]}^p f_{mn}^e. \end{aligned} \tag{6.48}$$

Condition (6.48) can be solved for β_1, but in the general case the single summands cannot be evaluated explicitly. We therefore turn to explicit examples in the following chapters.

7 Instantons on G_2-Structure Manifolds

We have seen in the previous chapters that the instanton equation on the cylinder over a general coset space G/H cannot be solved explicitly once the connection is generalized. The equations then contain sums that cannot be further simplified. We have to evaluate equations (6.40), (6.41), (6.47) and (6.48) on explicit coset spaces to find instanton solutions in generalized setups.

As explained in Chapter 4, coset spaces of dimensions five, six and seven with reduced structure group are of particular interest. Instantons on cones and cylinders over five-dimensional Sasakian manifolds have been discussed in [24] and [74]. We will present some non-BPS Yang-Mills solutions on these geometries in Chapter 9. Instantons on eight-dimensional cones and cylinders over seven-dimensional G_2-structure manifolds have been addressed in [28]. Seven-dimensional product spaces over a six-dimensional coset space with nearly-Kähler structure have been discussed in [24, 26, 28, 61, 63]. We choose to review this case to illustrate the meaning of our equations and investigate whether new instantons can be constructed when the nearly-Kähler structure is generalized. For this purpose, we will explicitly evaluate the instanton equation on the product space $\mathbb{R} \times SU(3)/(U(1) \times U(1))$, with the coset space admitting a half-flat $SU(3)$-structure.

Before considering the example, let us remark some general facts. The cylinder over an $SU(3)$-structure coset space comes with a torsionful G_2-structure. As described in Chapter 4.2, the restriction of the structure group to $G_2 \subset SO(7)$ uniquely determines a metric g_7, an orientation, a three-form P and hence a four-form $Q = *P$. The instanton equation (6.8) on the cylinder over a general coset space G/H turns into the following conditions, using the gauge connection

$\mathcal{A} = e^i I_i + e^a X_a$ and leaving the components of the four-form Q arbitrary[15]:

$$\dot{X}_a^b = \frac{1}{2} g^{cp} g^{dq} Q_{0apq} \left(f_{cd}^e X_e^b - X_c^m X_d^n f_{mn}^b \right), \tag{7.1}$$

$$0 = g^{cp} g^{dq} Q_{0apq} (f_{cd}^i - X_c^m X_d^n f_{mn}^i), \tag{7.2}$$

$$f_{ab}^i - X_a^m X_b^n f_{mn}^i = -\frac{1}{2} g^{cp} g^{dq} Q_{abpq} \left(f_{cd}^i - X_c^m X_d^n f_{mn}^i \right), \tag{7.3}$$

$$f_{ab}^e X_e^f - X_a^m X_b^n f_{mn}^f = \frac{1}{2} g^{cp} g^{dq} \left(g^{er} Q_{0eab} Q_{0rpq} - Q_{abpq} \right) \left(f_{cd}^e X_e^f - X_c^m X_d^n f_{mn}^f \right). \tag{7.4}$$

Clearly, these equations depend on the choice of Q. According to Chapter 3.2 of [64], the instanton equation on a non-compact G_2-structure manifold with four-form Q only admits a solution if Q is closed. Otherwise the system of first-order equations is overdetermined. The G_2-structure induced from the underlying $SU(3)$-structure is not unique and depends on the choice of structure form Q. The simplest closed, $SU(3)$-invariant four-form on the cylinder over a half-flat coset space that allows for a nonsingular G_2-structure metric is given by

$$Q = \frac{1}{2} \omega \wedge \omega + d\tau \wedge \Omega^-. \tag{7.5}$$

This form is closed due to equation (4.33) and $d\Omega^- = 0$. Inserting (7.5) into equation (4.58) shows that the standard cylinder metric $g = d\tau^2 + g_{G/H}$ is the correct G_2-structure metric corresponding to this four-form. This Q is the simplest possible four-form in the sense that without the $d\tau$-term in Q, the corresponding G_2-structure metric on the product manifold would become singular. As (7.5) is closed but not coclosed, it determines a cocalibrated G_2-structure on $\mathcal{Z}(G/H)$ with dual three-form $P = *_7 Q = \omega \wedge d\tau - \Omega^+$. This three-form has already been introduced for a nearly-Kähler coset space in Chapter 2.2 of [28].

Let us now specialize to $G/H = SU(3)/(U(1) \times U(1))$. Representation theoretic arguments and the $SU(3)$-invariance condition (3.33) allow us to further specify the matrix entries of the gauge connection $\mathcal{A} = e^i I_i + e^a X_a$ on $\mathbb{R} \times SU(3)/(U(1) \times U(1))$ with three complex-valued functions $\phi_1(\tau), \phi_2(\tau), \phi_3(\tau)$, as explained in detail in [26]:

$$\begin{aligned} X_1 &= Re(\phi_1) I_1 - Im(\phi_1) I_2, & X_4 &= -Im(\phi_2) I_3 + Re(\phi_2) I_4, \\ X_2 &= Im(\phi_1) I_1 + Re(\phi_1) I_2, & X_5 &= Re(\phi_3) I_5 - Im(\phi_3) I_6, \\ X_3 &= Re(\phi_2) I_3 + Im(\phi_2) I_4, & X_6 &= Im(\phi_3) I_5 + Re(\phi_3) I_6. \end{aligned} \tag{7.6}$$

[15] To avoid confusion, we emphasize that indices $a, b, c, d, e, f, m, n, p, q$ label directions on G/H and i labels directions on H in these equations.

Together with the four-form (7.5) and the corresponding G_2-structure metric

$$g_7 = d\tau^2 + R_1^2(e^1 \otimes e^1 + e^2 \otimes e^2) \\ +R_2^2(e^3 \otimes e^3 + e^4 \otimes e^4) + R_3^2(e^5 \otimes e^5 + e^6 \otimes e^6), \qquad (7.7)$$

the instanton conditions (7.1) to (7.4) yield the following set of three cyclic first-order and three algebraic relations that depend on the deformation parameters R_1, R_2, R_3:

$$\begin{aligned} \dot{\phi}_\alpha &= \frac{iR_\alpha}{\sqrt{3}R_\beta R_\gamma}\left(\bar{\phi}_\beta\bar{\phi}_\gamma - \phi_\alpha\right), \\ \frac{|\phi_\alpha|^2}{R_\alpha^2} - \frac{|\phi_\beta|^2}{R_\beta^2} &= \frac{1}{R_\alpha^2} - \frac{1}{R_\beta^2}, \end{aligned} \qquad \text{with } \alpha, \beta, \gamma \text{ cyclic.} \qquad (7.8)$$

Assuming that the deformation parameters are equal, $R_1 = R_2 = R_3$, implies $|\phi_1| = |\phi_2| = |\phi_3|$ and restricts the structure of the six-dimensional manifold to being nearly-Kähler. In this case, the algebraic condition is identically satisfied and the three differential equations coincide, taking the form

$$\dot{\phi} = \frac{i}{\sqrt{3}R}(\bar{\phi}^2 - \phi) \qquad (7.9)$$

with one complex function $\phi(\tau)$ and one real parameter R. This equation is similar to the instanton equation on the cocalibrated cylinder presented in [28, Chapter 5, equation (5.6 b)] with a suitable choice of R. The solution takes the form of a kink, as discussed in detail in Chapter 5 of [24], and is explicitly given by

$$\phi(\tau) = -\frac{1}{2}\left(1 + i\sqrt{3}\tanh\left(\frac{1}{2R}(\tau - \tau_0)\right)\right). \qquad (7.10)$$

Leaving the parameters R_1, R_2, R_3 arbitrary, the system of equations (7.8) has the potential to admit more general solutions with three different functions ϕ_1, ϕ_2, ϕ_3.

Equation (7.8) is a quadratic ordinary autonomous differential equation of first order. There are no general results about the existence of globally defined solutions (for all $\tau \in \mathbb{R}$) for systems of this type. Solutions to quadratic ODE's typically exhibit singularities at finite values of the variable τ.[16] The properties of the system (7.8) and its solutions are further investigated in Chapter 7.1.

The above instanton equation can be generalized as follows. The four-form (7.5) is not the most general choice on a restricted half-flat manifold. A direct

[16]This can be illustrated by studying the simple example of the ODE $\dot{x}(\tau) = x(\tau)^2$. A solution to this equation is given by $x(\tau) = \frac{1}{\tau - c}$ for some real constant $c \in \mathbb{R}$ and blows up at $\tau = c$.

computation shows that on $SU(3)/(U(1)\times U(1))$ with different R_1, R_2, R_3, the Hodge dual of the two-form ω is not proportional to the differential of the three-form $d\Omega$:

$$*\omega \neq d\Omega. \tag{7.11}$$

These forms have the same directions but different prefactors (different dependence on the deformation parameters). This suggests to write down another instanton equation with a four-form constructed as a linear combination of $*\omega$ and $d\Omega$. Equation (4.32) takes the following form on any half-flat coset space:

$$d\Omega^+ = -\mathcal{W}_1^+\omega\wedge\omega + \mathcal{W}_2^+\wedge\omega, \qquad d\Omega^- = 0. \tag{7.12}$$

We can therefore construct a closed four-form on the product space $\mathbb{R}\times G/H$ over a half-flat coset space G/H as

$$Q = e^{4\tau}\left(-\frac{1}{4}\mathcal{W}_1^+\omega\wedge\omega + \frac{1}{4}\mathcal{W}_2^+\wedge\omega + d\tau\wedge\Omega^+ + d\tau\wedge\Omega^-\right). \tag{7.13}$$

This does not have to be the most general closed, $SU(3)$-invariant four-form. A rescaling of the last summand with a real factor, for example, still leads to a closed form. For simplicity, we do not include this scaling factor in the following, as it will not significantly change our result.

As long as the G_2-structure metric corresponding to this four-form is not known, we cannot say anything about the geometry of the product space. Again, let us specialize to $\mathbb{R}\times G/H = \mathbb{R}\times SU(3)/(U(1)\times U(1))$, on which the metric can be computed with the method presented in Chapter 4.5. It turns out that the G_2-structure metric induced by the four-form (7.13) on $\mathbb{R}\times SU(3)/(U(1)\times U(1))$ is not the standard product one:

$$g_7 = e^{2\tau}4\cdot 3^{\frac{3}{4}}\sqrt{2R_1R_2R_3}\left(d\tau^2 + \frac{1}{24}\delta_{ab}e^ae^b\right). \tag{7.14}$$

Inserting Q and g into the instanton equations (7.1) to (7.4), we find the following cyclic conditions, which are similar to (7.8) but not explicitly R_α-dependent[17]:

$$\dot{\phi}_\alpha = 2(1-i)\left(\phi_\alpha - \bar{\phi}_\beta\bar{\phi}_\gamma\right), \qquad \alpha,\beta,\gamma \text{ cyclic.} \tag{7.15}$$

In addition, we find the algebraic relations

$$|\phi_1|^2 = |\phi_2|^2 = |\phi_3|^2. \tag{7.16}$$

[17] The different prefactors $\frac{1}{\sqrt{3}}$ and 2 on the right-hand sides of the differential equations (7.8) and (7.15) stem from the different prefactors of the components Q_{abcd} in (7.5) and (7.13).

Although the absolute values of all fields are forced to be equal, the parameters R_1, R_2, R_3 are left arbitrary, such that the instanton conditions do not restrict the geometry of the underlying half-flat six-dimensional space. Explicit solutions are expected to take a similar kink-type form as the solutions of equation (7.9).

To conclude, we have found two interesting four-forms Q that open the possibility for instanton solutions when the six-dimensional space is half-flat. The first four-form (7.5) allows to reproduce known solutions on the cylinder over $SU(3)/(U(1) \times U(1))$ with nearly-Kähler structure and possibly admits solutions with three different functions. In the second case with the four-form (7.13), the deformation parameters of the coset space contribute as one factor $\sqrt{R_1 R_2 R_3}$ in the seven-dimensional metric (7.14), hence from the seven-dimensional viewpoint it makes no difference whether the deformation parameters are equal or not. The solutions to the instanton equation are expected to take a similar kink-type form as in the nearly-Kähler case. The G_2-structure induced by both four-forms is cocalibrated, as Q is by construction closed. The seven-dimensional instantons that can be obtained from the second construction are therefore not new, although the underlying six-dimensional geometry is slightly more general than in earlier works.

Since Q is required to be closed, any construction of the presented type will lead to a cocalibrated G_2-structure. It would be interesting to study instantons on product spaces $\mathbb{R} \times G/H$ with other, possibly more general $SU(3)$-structure manifolds, for example $SU(2)\times SU(2)$. This space admits deformation parameters as well, as can be found in [49], and may allow for the construction of more than one four-form Q.

Let us close by noting that on G_2-structure seven-manifolds (and only on them), the instanton equation (6.1) is equivalent to the condition

$$Q \wedge \mathcal{F} = 0. \tag{7.17}$$

This equation does not depend explicitly on the metric. Using it instead of (6.14), the explicit computation of the metric can be avoided. The proof of equivalence of the instanton conditions is based on representation theoretic arguments and can be found in [29]. For instance, inserting the four-form (7.13) into this equation, we obtain the algebraic relation $\mathcal{F}_{12} - \mathcal{F}_{34} + \mathcal{F}_{56} = 0$ and the following differential equations:

$$-\frac{1}{2\sqrt{3}}\mathcal{F}_{01} + \mathcal{F}_{45} - \mathcal{F}_{46} - \mathcal{F}_{36} - \mathcal{F}_{35} = 0, \tag{7.18}$$
$$-\frac{1}{2\sqrt{3}}\mathcal{F}_{02} - \mathcal{F}_{46} - \mathcal{F}_{45} + \mathcal{F}_{36} - \mathcal{F}_{35} = 0, \tag{7.19}$$
$$\frac{1}{2\sqrt{3}}\mathcal{F}_{03} - \mathcal{F}_{25} + \mathcal{F}_{26} - \mathcal{F}_{16} - \mathcal{F}_{15} = 0, \tag{7.20}$$
$$\frac{1}{2\sqrt{3}}\mathcal{F}_{04} - \mathcal{F}_{26} - \mathcal{F}_{25} - \mathcal{F}_{16} + \mathcal{F}_{15} = 0, \tag{7.21}$$
$$-\frac{1}{2\sqrt{3}}\mathcal{F}_{05} - \mathcal{F}_{24} + \mathcal{F}_{14} - \mathcal{F}_{23} - \mathcal{F}_{13} = 0, \tag{7.22}$$
$$-\frac{1}{2\sqrt{3}}\mathcal{F}_{06} - \mathcal{F}_{24} - \mathcal{F}_{14} + \mathcal{F}_{23} - \mathcal{F}_{13} = 0. \tag{7.23}$$

Inserting the explicit expressions for $\mathcal{F}$ shows that these relations lead to equations (7.15) and (7.16).

7.1 On the Existence of Solutions to the Instanton Equation with Three Different Parameters R_1, R_2, R_3

For a better understanding of the ODE system (7.8) and the behaviour of its solutions, we first note that the algebraic constraints are compatible with the differential equations. They do not restrict the ODE system. To see this, we multiply the differential equation with $\bar{\phi}_\alpha$ and add the complex conjugate of this equation:

$$\dot{\phi}_\alpha\bar{\phi}_\alpha + \dot{\bar{\phi}}_\alpha\phi_\alpha = \frac{i}{\sqrt{3}}\frac{R_\alpha}{R_\beta R_\gamma}\left(\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3 - \phi_1\phi_2\phi_3\right). \tag{7.24}$$

This implies

$$\partial_\tau\left(\frac{|\phi_\alpha|^2}{R_\alpha^2}\right) = \frac{\dot{\phi}_\alpha\bar{\phi}_\alpha + \dot{\bar{\phi}}_\alpha\phi_\alpha}{R_\alpha^2} = \frac{i}{\sqrt{3}}\frac{1}{R_1R_2R_3}\left(\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3 - \phi_1\phi_2\phi_3\right). \tag{7.25}$$

The right-hand side of this equation takes the same form for all α, hence

$$\partial_\tau\left(\frac{|\phi_\alpha|^2}{R_\alpha^2} - \frac{|\phi_\beta|^2}{R_\beta^2}\right) = 0, \qquad \frac{|\phi_\alpha|^2}{R_\alpha^2} - \frac{|\phi_\beta|^2}{R_\beta^2} = const. \tag{7.26}$$

The constant is determined by the algebraic condition (7.8), which can be interpreted as an initial condition at the initial value $\tau_0 = 0$. Fixing, for example,

$|\phi_1(0)|^2 := p$ determines $|\phi_2(0)|^2$ and $|\phi_3(0)|^2$ as a function of the initial value $p \in \mathbb{R}$. Noting furthermore that the right-hand side of the differential equation (7.8) is locally Lipschitz continuous, and choosing initial angles $\theta_1(0)$, $\theta_2(0)$, $\theta_3(0)$, we find that the initial values uniquely determine a local solution to the system (7.8) with explicitly different deformation parameters $R_1 \neq R_2 \neq R_3$.

Global solutions are given by the equilibria

$$\dot{\phi}_1 = \dot{\phi}_2 = \dot{\phi}_3 = 0. \tag{7.27}$$

They are the constant solutions $\phi_1 = \phi_2 = \phi_3 = 0$ and

$$r_1 = r_2 = r_3 = 1, \qquad \theta_1 + \theta_2 + \theta_3 = 2\pi n, \tag{7.28}$$

where we have introduced polar coordinates $\phi_\alpha(\tau) = r_a(\tau)e^{i\theta_\alpha(\tau)}$.

Proving that the system allows for globally defined *dynamic* solutions can, as already mentioned, not be done in a standard way. For a closer investigation, it is useful to rescale the functions ϕ_α as

$$\phi_\alpha(\tau) \mapsto R_\alpha \phi_\alpha\left(\sqrt{3}\tau\right), \tag{7.29}$$

such that the ODE system (7.8) takes the form

$$\begin{aligned} \dot{\phi}_\alpha &= i\left(\bar{\phi}_\beta\bar{\phi}_\gamma - \frac{R_\alpha}{R_\beta R_\gamma}\phi_\alpha\right), \\ |\phi_\alpha|^2 - |\phi_\beta|^2 &= \frac{1}{R_\alpha^2} - \frac{1}{R_\beta^2}, \end{aligned} \qquad \text{with } \alpha,\beta,\gamma \text{ cyclic.} \tag{7.30}$$

Rewriting these equations in real coordinates $\phi_\alpha(\tau) := x_\alpha(\tau) + iy_\alpha(\tau)$ leads to the following conditions, cyclic in α, β, γ:

$$\begin{aligned} \dot{x}_\alpha &= x_\beta y_\gamma + x_\gamma y_\beta + \frac{R_\alpha}{R_\beta R_\gamma} y_\alpha = -\frac{\partial H}{\partial y_\alpha}, \\ \dot{y}_\alpha &= x_\beta x_\gamma - y_\beta y_\gamma - \frac{R_\alpha}{R_\beta R_\gamma} x_\alpha = \frac{\partial H}{\partial x_\alpha}, \\ x_\alpha^2 + y_\alpha^2 - x_\beta^2 - y_\beta^2 &= \frac{1}{R_\alpha^2} - \frac{1}{R_\beta^2}. \end{aligned} \tag{7.31}$$

These are six differential equations and three algebraic conditions. The differential equations constitute a Hamiltonian system with Hamiltonian function

$$H = x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3 - \frac{1}{2}\sum_{\alpha=1}^{3}\frac{R_\alpha}{R_\beta R_\gamma}(x_\alpha^2 + y_\alpha^2). \tag{7.32}$$

Hamiltonian systems admit statements about the existence of global solutions in the following way. As can be found for example in [65, Example 1.6.8] or [66, Proposition 8.10], a Hamiltonian system admits a global solution (defined for all $\tau \in \mathbb{R}$) if

$$\lim_{\|(x_1,x_2,x_3,y_1,y_2,y_3)\|\to\infty} H(x_1, x_2, x_3, y_1, y_2, y_3) = \infty, \tag{7.33}$$

with $\|\cdot\|$ denoting the standard norm in $\mathbb{R}^6$. A direct computation shows that our Hamiltonian (7.32) does not satisfy this property. We consider the case $x_2 = x_3 = 0$, $x_1 = -\frac{1}{2}\frac{1}{R_1R_2R_3}(R_1^2 + R_2^2 + R_3^2) < \infty$ and find

$$\begin{aligned}
&\lim_{\|y_\alpha\|\to\infty} H(x_1, 0, 0, y_1, y_2, y_3) \\
&\quad = \lim_{\|y_\alpha\|\to\infty}\left(-x_1y_2y_3 - \frac{1}{2}\left(\frac{R_1}{R_2R_3}(x_1^2 + y_1^2) + \frac{R_2}{R_1R_3}y_2^2 + \frac{R_3}{R_1R_2}y_3^2\right)\right) \\
&\quad = -\frac{1}{8}\frac{1}{R_2^2R_3^2}\left(R_1^2 + R_2^2 + R_3^2\right) < \infty.
\end{aligned} \tag{7.34}$$

This shows that in our case no definite statement about global existence of solutions can be made in this way.

Another possibility to show the existence of global solutions is a closer study of the phase space of the system (7.31). Solutions of a Hamiltonian system are located on the level sets of the Hamiltonian function. The existence of bounded (compact) level sets of the Hamiltonian (7.32) in the phase space $\mathbb{R}^6$ would guarantee the existence of global solutions. The idea can be best illustrated in the case of a two-dimensional phase space, i. e. for $R_1 = R_2 = R_3$ and $\phi_1 = \phi_2 = \phi_3$. The level sets of the corresponding Hamiltonian

$$H = x^3 - 3xy^2 - \frac{3}{2}\frac{1}{R}\left(x^2 + y^2\right) \tag{7.35}$$

are shown in Figure 2 for the case $R = 1$. The phase space is cut into seven distinct regions by the constant solution $H(-\frac{1}{2}, y) = -\frac{1}{2}$. This solution can be rotated by 3-symmetry, giving rise to the three intersecting straight lines visible in the plot. These lines border a compact subset of the phase space, ensuring the existence of globally defined, bounded solutions inside this set and suggesting divergent solutions outside, possibly with blow-up behaviour.

It turns out that compact subsets of the phase space of the six-dimensional system cannot be found in such a straightforward way. The two-dimensional phase space allows for one-dimensional straight hypersurfaces after fixing one of the coordinates. The six-dimensional Hamiltonian (7.32) does however not obviously

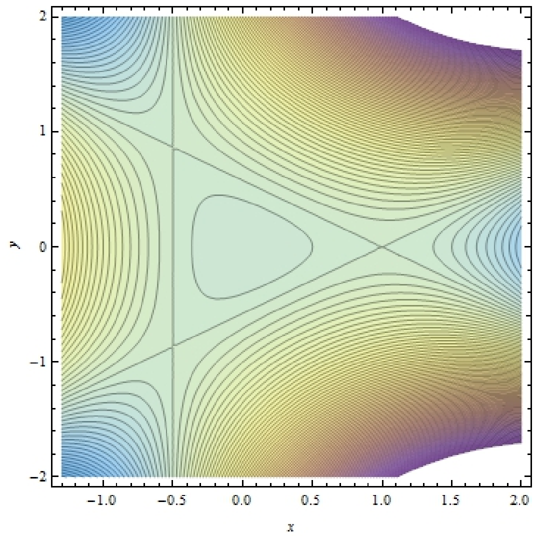

Figure 2: Level sets of the Hamiltonian (7.35) with parameter value $R = 1$. The plot shows the straight lines on which the analytic solution (7.10) is located as well as bounded and unbounded level sets. Bounded solutions are located on the closed level sets of H in the center of the plot.

admit five-dimensional hypersurfaces with one or more fixed coordinates because each of the six phase space coordinates appears both in linear and in quadratic order. Furthermore, as H is a polynomial of degree 2 in each coordinate, we cannot expect the $H = const$ hypersurfaces to be bounded in any direction of phase space. In general, statements about the phase space of six-dimensional Hamiltonian systems are hard to make.

Future work for understanding the behaviour of solutions of (7.8) may include the study of chaotic behaviour, which appears frequently in nonlinear Hamiltonian systems of dimensions higher than two. This investigation, however, is beyond the scope of this thesis.

Part III

Non-BPS Yang-Mills Solutions on Coset Spaces

8 Yang-Mills Equation on the Cone over a Coset Space

In Chapter 6, the higher-dimensional instanton equation has been written out in components and solved on the product space $\mathbb{R}\times G/H$, using the simplified ansatz $\mathcal{A} = e^i I_i + \phi e^a I_a$ for the gauge connection. In this chapter, we discuss the Yang-Mills equation in an analogous setup. As the instanton equation is conformally invariant, it was sufficient to consider the cylinder over G/H to obtain results that also hold on the cone. This is not true for the Yang-Mills equation, hence cone and cylinder have to be discussed separately now.

For a motivation of the following discussion, recall that a connection with totally antisymmetric torsion naturally appears in the conditions for supersymmetry preservation in heterotic supergravity [36]. On suitably chosen string backgrounds, one can introduce geometric three-form fluxes that are identified with the torsion of this spin connection. The Yang-Mills equation with torsion on a Riemannian manifold takes the following form with some three-form $\mathcal{H}$:

$$d * \mathcal{F} + [\mathcal{A}, *\mathcal{F}] + *\mathcal{H} \wedge \mathcal{F} = 0. \tag{8.1}$$

Recall furthermore that the Yang-Mills equation follows from covariant differentiation of the higher-dimensional instanton equation if the three-form is related to the G-structure four-form as $*\mathcal{H} := d * Q$. In this case, the Yang-Mills equation is the equation of motion of the action (5.10) consisting of a Yang-Mills plus a Chern-Simons term. The torsion term in the Yang-Mills equation is generated by variation of the Chern-Simons term, while the other summands arise from variation of the Yang-Mills term.

Non-BPS Yang-Mills solutions can be constructed when the Yang-Mills equation is not required to follow from a first-order equation. In accordance with earlier work [23, 24, 26], we choose to identify the three-form $\mathcal{H}$ with the torsion of the spin connection, $\mathcal{H}_{ABC} \propto T_{ABC}$, and the torsion components[18] to be proportional to the structure constants on G/H:

$$T_{abc} = \kappa f_{abc}, \qquad\qquad \kappa \in \mathbb{R}. \tag{8.2}$$

In explicit examples, the relation of T and $\mathcal{H}$ will be chosen such that $*\mathcal{H} = d * Q$ is satisfied for $\kappa = 1$ and the Yang-Mills equation follows from the instanton equation for this value of κ. Other choices are possible and correspond to a rescaling of the parameter κ. Solutions of the torsionful Yang-Mills equation can be lifted to solutions of heterotic supergravity if they follow from a first-order BPS equation. More general non-BPS Yang-Mills solutions for arbitrary values of κ can potentially serve as building blocks for non-supersymmetric string solutions.

Written out in components, the torsionful Yang-Mills equation on the product space $\mathbb{R} \times G/H$ turns into the following set of equations (see Appendix C.2 for a detailed derivation), where the metric is assumed to be of diagonal form with coordinate-dependent components:

$$\begin{aligned} &\frac{g_{BB}}{\sqrt{|g|}}\partial_C\left(\sqrt{|g|}\mathcal{F}^{CB}\right) \\ &\qquad - \mathcal{F}^{CD}\left(\frac{1}{2}T_{CDB} - {}^{-}\Gamma_{CDB}\right) + \mathcal{F}^{C}{}_{B}\left(\frac{1}{2}T_{CD}{}^{D} - {}^{-}\Gamma_{CD}{}^{D}\right) \\ &\qquad - \mathcal{F}^{C}{}_{B}\left(\frac{1}{2}T_{DC}{}^{D} - {}^{-}\Gamma_{DC}{}^{D}\right) + [\mathcal{A}^A, F_{AB}] - \frac{1}{2}\mathcal{H}_{CDB}\mathcal{F}^{CD} = 0. \end{aligned} \tag{8.3}$$

These are d equations, with the free index B running from 0 to $\dim G/H$. The coefficients $\mathcal{H}_{ABC}$ (with all indices lowered) are the components of the 3-form $\mathcal{H}$, and ${}^{-}\Gamma^C_{AB}$ are the coefficients of the torsionful spin connection with torsion T^C_{AB}. This equation[19] has been discussed in detail on the cylinder over an arbitrary coset

[18] It has been argued in [24] that for such a choice of $\mathcal{H}_{ABC}$ and T_{ABC}, the Yang-Mills equation on the cylinder over a nearly-Kähler coset space follows from an action similar to (5.10). This does not have to hold for other choices of $\mathcal{H}$.

[19] Note that this equation is not identical to the corresponding equations (2.19) and (2.20) in

space G/H with $\mathcal{A} = e^i I_i + \phi e^a I_a$ in [23, 24, 26, 75], leading to explicit kink-type solutions of similar type as those presented in Figure 1.

Equation (8.3) turns into the following equations on the cone $\mathbb{R} \times G/H$ with metric $g_{\mathcal{C}} = \gamma^2 e^{2\gamma\tau}(d\tau^2 + \delta_{ab}e^a e^b)$, where γ denotes the opening angle:

$$\mathcal{F}^{CD}\left(\frac{1}{2}T_{CD}{}^0 - {}^-\Gamma_{CD}{}^0\right) - \mathcal{F}^{c0}\left(\frac{1}{2}T_{cD}{}^D - {}^-\Gamma_{cD}{}^D\right) + \mathcal{F}^{c0}\left(\frac{1}{2}T_{Dc}{}^D - {}^-\Gamma_{Dc}{}^D\right) - [\mathcal{A}_a, F^{a0}] + \frac{1}{2}\mathcal{H}_{CD}{}^0\mathcal{F}^{CD} = 0 \quad (8.4)$$

for $B = 0$ and

$$\gamma^{-4}e^{-4\gamma\tau}\partial_0\mathcal{F}_{0b} + \gamma(d-4)\mathcal{F}^{0b} - \mathcal{F}^{CD}\left(\frac{1}{2}T_{CD}{}^b - {}^-\Gamma_{CD}{}^b\right) + \mathcal{F}^{Cb}\left(\frac{1}{2}T_{CD}{}^D - {}^-\Gamma_{CD}{}^D\right) - \mathcal{F}^{Cb}\left(\frac{1}{2}T_{DC}{}^D - {}^-\Gamma_{DC}{}^D\right) + [\mathcal{A}_a, F^{ab}] - \frac{1}{2}\mathcal{H}_{CD}{}^b\mathcal{F}^{CD} = 0 \quad (8.5)$$

for $B \neq 0$.

Some of the connection coefficients ${}^-\Gamma^C_{AB}$ vanish on the cylinder but not on the cone. They are derived by use of Cartan's structure equation (3.25) and metric compatibility (3.24), employing the ansatz (8.2) for the torsion. Apart from the coefficients

$${}^-\Gamma^c_{ab} = \frac{1}{2}(\kappa + 1)f^c_{ab} + f^c_{ib}e^i_a, \quad (8.6)$$

which have been derived in [75] and are the same on the cone and on the cylinder, we find the following nonvanishing components on the cone:

$${}^-\Gamma^0_{00} = 1, \quad (8.7)$$
$${}^-\Gamma^0_{ab} = {}^-\Gamma^0_{ba} = -\delta_{ab}, \quad (8.8)$$
$${}^-\Gamma^a_{b0} = {}^-\Gamma^a_{0b} = \delta_{ab}. \quad (8.9)$$

The spin connection then has the following nonvanishing coefficients:

$${}^-\Gamma^c_b = {}^-\Gamma^c_{Ab}e^A, = (1 + \delta_{bc})\, e^0 + \frac{1}{2}(\kappa + 1)f^c_{ab}e^a + f^c_{ib}e^i, \quad (8.10)$$
$${}^-\Gamma^c_0 = {}^-\Gamma^c_{a0}e^a = \delta_{ac}e^a, \quad (8.11)$$
$${}^-\Gamma^0_b = {}^-\Gamma^0_{ab}e^a = -\delta_{ab}e^a. \quad (8.12)$$

[26] due to differently normalized torsion. The equations presented in the reference follow from our equation (8.3) with cylinder metric $g_{\mathcal{Z}} = d\tau^2 + \delta_{ab}e^a e^b$ in the special case of $H_{ABC} = -T_{ABC}$.

For further specification, we have to fix the relation between the torsion T_{AB}^{C} and the three-form $\mathcal{H}$. As mentioned above, $H_{ABC} \propto T_{ABC}$ is chosen such that the relation $*\mathcal{H} = d * Q$ holds for a certain value of κ (so that the instanton case can be recovered for this κ-value). In the case of general G/H, there is some freedom in the choice of the structure form Q, and therefore also in the choice of $\mathcal{H}$. For this reason, and to avoid unnecessary complications in the following computation, we choose

$$\mathcal{H}_{abc} = -T_{abc} = -\kappa f_{abc}. \tag{8.13}$$

With this choice, our Yang-Mills equation (8.3) takes the same form as in [23, 26], so that the results can be directly compared. In particular, the torsion terms in the Yang-Mills equation cancel, and equation (8.5) with index $B \neq 0$ turns into

$$\gamma^{-4} e^{-4\gamma\tau} \partial_0 \mathcal{F}_{0b} + \gamma(d-4)\mathcal{F}^{0b} + \mathcal{F}^{cd}\ {}^{-}\Gamma^{b}_{cd} + \mathcal{F}^{cb}\ {}^{-}\Gamma^{d}_{dc} + \left[\mathcal{A}_a, \mathcal{F}^{ab}\right] = 0. \tag{8.14}$$

Note that all contributions from connection coefficients with zero index cancel, and the only difference to the corresponding cylinder equation is the first-order friction term. We insert the most general G-equivariant ansatz for the gauge connection,

$$\mathcal{A} = e^i I_i + e^a X_a, \tag{8.15}$$

with curvature

$$\mathcal{F}_{0a} = \dot{X}_a, \tag{8.16}$$

$$\mathcal{F}_{bc} = -(f^{i}_{bc} I_i + f^{a}_{bc} X_a - [X_b, X_c]), \tag{8.17}$$

and find a set of $(d-1)$ differential equations from the $B \neq 0$ equations[20] (8.5):

$$\ddot{X}_b + \gamma(d-4)\dot{X}_b + \frac{1}{2}\left(1 - \alpha(\kappa+2)\right) X_b + \frac{1}{2}(\kappa+3) f^{b}_{ad}[X_a, X_d] + [X_a, [X_a, X_b]] = 0. \tag{8.18}$$

The $B = 0$ equation (8.4) turns into the Gauss-law constraint, which is equivalent on the cone and on the cylinder:

$$[X_a, \dot{X}_a] = 0 \qquad \text{(sum over } a\text{)}. \tag{8.19}$$

The Yang-Mills equation with this ansatz contains terms proportional to the functions e^i_a, which cancel by use of the Jacobi identity (C.2) and G-invariance (3.33).

[20] Note that the friction term vanishes in dimension $d = 4$ and the Yang-Mills equation is conformally invariant in that dimension.

This has been shown in [76], hence these terms are not explicitly displayed here. Specializing to a gauge connection with one scalar field,

$$\mathcal{A} = e^i I_i + \phi(\tau) e^a I_a, \tag{8.20}$$

equation (8.19) is identically satisfied and equations (8.18) reduce to

$$\ddot{\phi} + \gamma(d-4)\dot{\phi} - \frac{1}{2}(\alpha+1)\phi(\phi-1)(\phi-\rho) = 0, \tag{8.21}$$

using the abbreviation

$$\rho := \frac{\alpha(\kappa+2)-1}{\alpha+1}. \tag{8.22}$$

In the following, we present solutions to this second-order equation on the cone. Equation (8.21) is also known as Duffing-Helmholtz equation. We will close this chapter with a discussion of this equation and a presentation of known solutions.

8.1 Solutions to the Yang-Mills Equation

The second-order equation (8.21) takes the following form after rescaling

$$\tau \mapsto \sqrt{\frac{\alpha+1}{2}}\tau \tag{8.23}$$

and introducing the potential $V(\phi)$:

$$\ddot{\phi} + \gamma(d-4)\sqrt{\frac{2}{\alpha+1}}\dot{\phi} = \phi(\phi-1)(\phi-\rho) := \frac{dV}{d\phi}. \tag{8.24}$$

To construct first-order equations that imply (8.24), we assume that the second-order equation follows from a flow equation[21] of the form

$$\dot{\phi} = \frac{dW(\phi)}{d\phi}, \tag{8.25}$$

with prepotential $W(\phi)$. Taking the τ-derivative and inserting $\dot{\phi}$ back yields

$$\ddot{\phi} = \frac{d^2W(\phi)}{d\phi^2} \cdot \frac{dW(\phi)}{d\phi}. \tag{8.26}$$

[21]This is known as gradient flow or Hamiltonian flow, depending on whether the proportionality $\dot{\phi} \propto \frac{\partial W}{\partial \phi}$ is real or imaginary. We will find gradient flow equations only.

These expressions can be inserted into the original second-order equation (8.24) to obtain

$$\frac{d^2W(\phi)}{d\phi^2}\cdot\frac{dW(\phi)}{d\phi}+\gamma(d-4)\sqrt{\frac{2}{\alpha+1}}\frac{dW(\phi)}{d\phi}=\frac{dV(\phi)}{d\phi}. \tag{8.27}$$

Integration with respect to ϕ yields

$$V(\phi)=\gamma(d-4)\sqrt{\frac{2}{\alpha+1}}W(\phi)+\frac{1}{2}\left(\frac{dW(\phi)}{d\phi}\right)^2+const. \tag{8.28}$$

According to its definition in equation (8.24), the potential $V(\phi)$ is a polynomial of degree four in ϕ and can be written as

$$V(\phi)=\frac{1}{4}\phi^4-\frac{1}{3}(1+\rho)\phi^3+\frac{1}{2}\rho\phi^2+const. \tag{8.29}$$

Assuming that W is of the form

$$W(\phi)=a\phi^3+b\phi^2+c\phi+p \tag{8.30}$$

with real coefficients a,b,c,p, and inserting this into equation (8.28) yields

$$\begin{aligned}V(\phi)=\frac{9}{2}a^2\phi^4+a\left(\Theta+6b\right)\phi^3+\left(3ac+2b^2+\Theta b\right)\phi^2&\\+c\left(\Theta+2b\right)\phi+\Theta p+\frac{1}{2}c^2,&\end{aligned} \tag{8.31}$$

where we have introduced the abbreviation $\Theta:=\gamma(d-4)\sqrt{\frac{2}{\alpha+1}}$. Comparing the coefficients of equation (8.29) and (8.31) leads to the following conditions on a,b,c and Θ:

$$\frac{1}{4}=\frac{9}{2}a^2 \qquad \text{for } \phi^4, \tag{8.32a}$$

$$-\frac{1}{3}(1+\rho)=a\left(\Theta+6b\right) \qquad \text{for } \phi^3, \tag{8.32b}$$

$$\frac{1}{2}\rho=\Theta b+2b^2+3ac \qquad \text{for } \phi^2, \tag{8.32c}$$

$$c\left(\Theta+2b\right)=0 \qquad \text{for } \phi. \tag{8.32d}$$

This system of equations leads to six solutions on the parameters a,b and c, each of them with one additional condition that relates ρ,γ and α.

The constant term p in the prepotential W, as well as the constant term in the potential V, remain undetermined in this computation. The constant can be determined by explicitly computing the action that has (8.24) as equation of motion. This can be done in a similar way as presented in Appendix C.3 for the cylinder over a Sasakian manifold. The action contains a kinetic term $\propto \dot{\phi}^2$ and the full potential. Knowing the full potential, in particular the constant term, allows for a statement about finiteness of the action as discussed earlier. We will not further discuss this point here.

For the construction of explicit solutions to (8.24), it is not necessary to know the constant term in V. From the system (8.32), we find the following first-order equations that solve the second-order equation:

$$\pm\dot{\phi} = \frac{1}{\sqrt{2}}\phi(\phi-\rho) \qquad\qquad \rho = 2 \pm 2\frac{\gamma(d-4)}{\sqrt{\alpha+1}}, \tag{8.33a}$$

$$\pm\dot{\phi} = \frac{1}{\sqrt{2}}\phi(\phi-1) \qquad \text{along with} \qquad \rho = \frac{1}{2} \mp \frac{\gamma(d-4)}{\sqrt{\alpha+1}}, \tag{8.33b}$$

$$\pm\dot{\phi} = \frac{1}{\sqrt{2}}(\phi-1)(\phi-\rho) \qquad\qquad \rho = \pm 2\frac{\gamma(d-4)}{\sqrt{\alpha+1}} - 1. \tag{8.33c}$$

Each solution comes with an additional condition on the parameters ρ, γ and α, which, after inserting equation (8.22), turns into the following relations between the torsion parameter κ, the cone dimension d, the opening angle γ and the coset parameter α. For the respective cases:

$$\kappa = \frac{1}{\alpha}\left(3 \pm 2\gamma(d-4)\sqrt{\alpha+1}\right), \tag{8.34a}$$

$$\kappa = \frac{1}{\alpha}\left(\frac{3}{2}(1-\alpha) \mp \gamma(d-4)\sqrt{\alpha+1}\right), \tag{8.34b}$$

$$\kappa = -3 \pm 2\gamma(d-4)\frac{\sqrt{\alpha+1}}{\alpha}. \tag{8.34c}$$

This is in agreement with the fact that the second-order equation follows from a first-order equation only for certain values of κ. Given that α and d are fixed on specific coset spaces, these relations allow to express κ as a function of γ and vice versa.

Equations (8.33a) and (8.33b) are equivalent for $\rho = 1$, and equations (8.33b) and (8.33c) are equivalent for $\rho = 0$. All explicit solutions of the three equations are of kink-type and take the respective form

$$\phi_1(\tau) = \pm\frac{\rho}{2}\tanh\left(\frac{\rho}{2\sqrt{2}}(\tau-\tau_0)\right) + \frac{\rho}{2}, \tag{8.35a}$$

$$\phi_2(\tau) = \pm\frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}(\tau-\tau_0)\right) + \frac{1}{2}, \tag{8.35b}$$

$$\phi_3(\tau) = \pm\frac{\rho-1}{2}\tanh\left(\frac{1-\rho}{2\sqrt{2}}(\tau-\tau_0)\right) + \frac{\rho+1}{2}. \tag{8.35c}$$

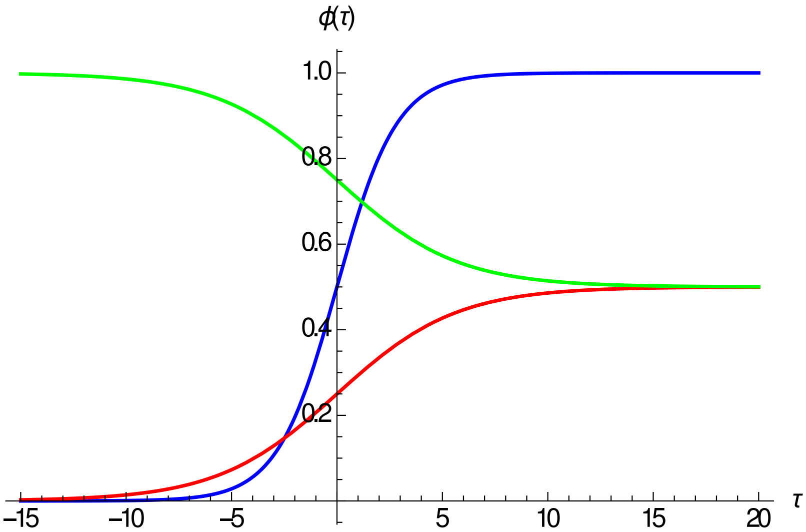

Figure 3: Plot of the solutions $\phi_1(\tau)$ (red), $\phi_2(\tau)$ (blue) and $\phi_3(\tau)$ (green) as in equations (8.35) with positive sign. The parameters have been chosen as $\tau_0 = 0$, $\alpha = \frac{1}{3}$ and $\kappa = 3$, implying $\rho = \frac{1}{2}$.

For positive signs on the right-hand sides, we find

$$\lim_{\tau\to-\infty} \phi_1(\tau) = 0 \qquad \lim_{\tau\to\infty} \phi_1(\tau) = \rho, \tag{8.36a}$$

$$\lim_{\tau\to-\infty} \phi_2(\tau) = 0 \qquad \lim_{\tau\to\infty} \phi_2(\tau) = 1, \tag{8.36b}$$

$$\lim_{\tau\to-\infty} \phi_3(\tau) = 1 \qquad \lim_{\tau\to\infty} \phi_3(\tau) = \rho, \tag{8.36c}$$

(the asymptotic values are exchanged for negative overall signs). Each of the constructed solutions interpolates between two of the three critical points ϕ_{crit} that satisfy $\frac{\partial V}{\partial \phi} = 0$. A plot for all three cases with a certain value of ρ can be found in Figure 3. Note that equation (8.35b) is equivalent to the instanton equation (6.34) after a suitable rescaling of τ. The instanton equation with Q constructed as in Chapter 6.1 therefore implies the Yang-Mills equation, but it is not the only solution to the second-order equation (8.24).

Some further steps can be taken to better understanding the constructed solutions. As mentioned, an explicit computation of the action can determine the constant term in the potential V. In the absence of the friction term in the Yang-Mills equation (8.24), finite-action Yang-Mills solutions must interpolate between critical points with zero potential, i. e. points that satisfy $\frac{\partial V}{\partial \phi} = V = 0$. Typically, the critical points satisfy $V = 0$ only for a certain value of κ. Including the friction term, the situation becomes more complicated, but the finite-action requirement is still expected to single out certain values of κ. Together with the conditions

(8.34), this analysis allows to determine a distinguished opening angle for which all described conditions are satisfied. We leave this task for future work.

8.2 The Duffing-Helmholtz Equation

Equations of the type (8.21) have been studied before, for example in [77, 78]. Equation (8.21) is known as damped Duffing-Helmholtz equation, which takes the general form

$$\ddot{x}(\tau) + 2\nu\dot{x}(\tau) + Ax(\tau) + Bx^2(\tau) + \varepsilon x^3(\tau) = 0, \tag{8.37}$$

with $x(\tau)$ being some (unknown) function, the dots denoting derivatives with respect to τ and the coefficients named in accordance with [77]. This equation models, for example, the dynamics of thin laminated plates, graded beams and eardrum oscillators [79–81].

In the special case of $B = 0$ (i. e. vanishing of the quadratic term, which corresponds to $\rho = -1$ or equivalently $\kappa = -3$ in equation (8.21)), we obtain the force-free, damped Duffing equation

$$\ddot{x}(\tau) + 2\nu\dot{x}(\tau) + Ax(\tau) + \varepsilon x^3(\tau) = 0. \tag{8.38}$$

This equation has first been introduced by Georg Duffing in 1918 [82] and is used to describe certain damped and driven oscillators, for example a spring pendulum with a special kind of spring stiffness (see [78] for more applications). A general solution of the damped Duffing equation is given by

$$x(\tau) = a(\tau)\ \mathrm{cn}(\omega(\tau), k^2), \tag{8.39}$$

provided that the parameters satisfy the condition [83]

$$A = \frac{8\nu^2}{9}, \tag{8.40}$$

where $\mathrm{cn}(\omega(\tau), k^2)$ denotes the Jacobi elliptic function, ω is the (time-dependent) frequency and k the elliptic modulus. In our case with $\kappa = -3$, the parameter condition (8.40) turns into the following constraint on the opening angle:

$$\gamma = \pm\sqrt{\frac{9(\alpha+1)}{8(d-4)^2}}. \tag{8.41}$$

As we are interested in the case of general κ, γ, we do not further discuss equation (8.38) and its solutions in detail here and instead turn back to the more general Duffing-Helmholtz equation (8.37).

Provided that the parameters satisfy the condition [77]

$$A = \frac{3B12 + 8\varepsilon\nu^2}{9\varepsilon}, \tag{8.42}$$

a solution to equation (8.37) is given by

$$x(\tau) = \frac{a(\tau) - b(\tau) + c(\tau)(a(\tau) - b(\tau))\mathrm{cn}(\omega(\tau), k^2)}{1 + c(\tau)\mathrm{cn}(\omega(\tau), k^2)}, \tag{8.43}$$

with

$$a(\tau) = -\frac{b(\tau)(c^4+1)}{c^4-1} - \frac{B}{3\varepsilon}, \tag{8.44}$$

$$b(\tau) = B_1 e^{-\frac{2}{3}\nu\tau}, \tag{8.45}$$

$$\omega(\tau) = -\frac{3A_1B_1}{2\nu} e^{-\frac{2}{3}\nu\tau} + C_0, \tag{8.46}$$

where A_1 and B_1 are integration constants that are determined by the initial conditions. The constant parameter c can be determined by a direct computation as described in [77].

With our parameters, we find that condition (8.42) turns into

$$\rho = \frac{1}{2} \pm \sqrt{\frac{16\gamma^2(d-4)^2 - 9(\alpha+1)}{12(\alpha+1)}}, \tag{8.47}$$

again constituting an additional constraint on the parameters ρ and γ (or, more precisely, on κ and γ). Note that this condition matches none of the cases (8.33), hence the solution (8.43) does not coincide with any of the ones constructed in Chapter 8.1.

9 Yang-Mills Equation on Cylinders over Sasakian Manifolds

Let us return to the Yang-Mills equation on the cylinder and consider the example in which the base manifold M has Sasakian structure and is of dimension $2m+1$. The cases $m = 2$ and $m = 3$ are of particular interest in string theory. The results of the following chapters can also be found in [84].

As introduced in Chapter 4.3, Sasakian manifolds have $U(m)$ structure group, while the structure group on Sasaki-Einstein manifolds reduces further to $SU(m)$ (see also [47]). We consider a Sasakian manifold M with metric

$$g_M = e^1 e^1 + e^{2h} \delta_{ab} e^a e^b, \tag{9.1}$$

which is not Einstein for arbitrary $h \in \mathbb{R}$. However, this construction still allows for an $SU(m)$ structure on M. To understand this, first note that there is no one-to-one correspondence of the existence of an $SU(m)$ structure on M and the Einstein property of the metric. If a manifold is Sasaki-Einstein, it must have $SU(m)$ structure group, but the converse is not necessarily true.

The structure of our manifold arises as follows. Starting with a Sasaki-Einstein manifold that has structure group $SU(m)$ and admits two Killing spinors[22] ϵ, $\tilde{\epsilon}$, one may construct the canonical connection ${}^P\nabla$ by the requirement ${}^P\nabla\epsilon = {}^P\nabla\tilde{\epsilon} = 0$. ${}^P\nabla$ has holonomy $SU(m)$ and components (4.18), (4.19). One then notices that ${}^P\nabla$ is compatible with the whole family of metrics (4.20)[23]. Deformation of the metric therefore does not affect the spinor identities ${}^P\nabla\epsilon = {}^P\nabla\tilde{\epsilon} = 0$, although the

[22] The Killing spinors can be used to construct the contact form η and the other structure forms, as described in [29].

[23] The compatibility can be verified by explicitly computing ${}^P\nabla g_h$, using g_h to raise and lower indices, or by rewriting g_h in terms of e^1 and $g_{h=0}$ and employing ${}^P\nabla e^1 = 0$ as well as ${}^P\nabla g_{h=0} = 0$. We thank Derek Harland for this comment.

connection ${}^P\nabla$ depends on the choice of the metric. The existence of two Killing spinors, on the other hand, is in one-to-one correspondence with the existence of an $SU(m)$-structure. Hence, the family g_h preserves the $SU(m)$ structure although the Einstein property is lost. We find that we have a Sasakian manifold with $SU(m)$-structure that is explicitly not Einstein .

As described in Chapter 4.3, we will restrict the consideration to a metric that makes the torsion of the canonical connection totally antisymmetric. The metric on the cylinder $\mathcal{Z}(M)$ then takes the form

$$g = e^0 e^0 + e^1 e^1 + \frac{2m}{m+1}\delta_{ab} e^a e^b, \tag{9.2}$$

where $e^0 := d\tau$ denotes the coordinate in the $\mathbb{R}$-direction. As the base manifold admits an $SU(m)$ structure and according to [29], we can construct a connection that has holonomy group $SU(m+1)$ on the cylinder. This corresponds to having an $SU(m+1)$-structure on $\mathcal{Z}(M)$ and, as described in Chapter 3, to a principal bundle $P(\mathcal{Z}(M), SU(m+1))$. We denote the gauge connection in this bundle by $\mathcal{A}$ and construct the connection induced by $\mathcal{A}$ in the tangent bundle over $\mathcal{Z}(M)$ as a generalization of the canonical connection. The Sasakian manifold M has structure group $SU(m)$ as described, and the canonical connection on M lifts to a connection on $T\mathcal{Z}(M)$ with the same holonomy group $SU(m) = Hol({}^P\nabla)$ as on the base space. The existence of an $SU(m)$-holonomy connection in $T\mathcal{Z}(M)$ corresponds to the existence of a principal subbundle of P with structure group $SU(m)$.

As $SU(m)$ is a subgroup of $SU(m+1)$, the corresponding Lie algebras split according to $\mathfrak{su}(m+1) = \mathfrak{su}(m) \oplus \mathfrak{m}$ with respect to the Killing metric, where $SU(m)$ acts irreducibly on $\mathfrak{su}(m)$ and $\mathfrak{m}$ denotes the $(2m{+}1)$-dimensional orthogonal complement to $\mathfrak{su}(m)$. We denote the $\mathfrak{su}(m+1)$-generators that span the orthogonal complement $\mathfrak{m}$ as $\{I_\mu\} = \{I_1, I_a\}$ with indices as in Chapter 4.3, and the remaining generators of $\mathfrak{su}(m)$ as $\{I_i\}$, in analogy to the splitting on a reductive coset space. The groups considered here, however, do not necessarily have to form a quotient. The $SU(m+1)$-generators satisfy the commutation relations

$$\begin{aligned} [I_i, I_j] &= f^k_{ij} I_k, \\ [I_i, I_\mu] &= f^\nu_{i\mu} I_\nu, \\ [I_\mu, I_\nu] &= f^\rho_{\mu\nu} I_\rho + f^k_{\mu\nu} I_k. \end{aligned} \tag{9.3}$$

Written as $SU(m+1)$-matrices, they have the following nonvanishing entries:

$$\begin{aligned} I^b_{ia} &= f^b_{ia}, \\ I^b_{1a} &= -\frac{1}{m}\omega_{ab}, & -I^0_{11} &= I^1_{10} = 1, \\ -I^0_{ab} &= I^b_{a0} = \delta^b_a, & -I^1_{ab} &= I^b_{a1} = \omega_{ab}. \end{aligned} \tag{9.4}$$

In this basis, the $SU(m)$-structure constants are related as follows to the $SU(m)$-structure three-form on the base manifold M:

$$f^1_{ab} = 2P_{ab1}, \qquad f^b_{1a} = \frac{m+1}{m}P_{1ab}. \tag{9.5}$$

As in Chapter 3, we make the following ansatz for the gauge connection on $T\mathcal{Z}(M)$:

$$\mathcal{A} = {}^P\nabla + X_\mu e^\mu. \tag{9.6}$$

By construction, the components $X_\mu e^\mu$ are (local) representations of a globally defined connection one-form X subject to the conditions (3.2) on the principal gauge bundle $P(SU(m+1))$. In general, they depend on the choice of local coframe $\{e^\mu\}$. Requiring them to be independent under a change of coframe leads to the following invariance condition:

$$[I_i, X_\mu] = f^\nu_{i\mu} X_\nu. \tag{9.7}$$

For a more detailed motivation in this context, we also refer to [74]. This condition takes the same form as condition (3.33) for G-invariant connections on coset spaces G/H, but the construction on coset spaces is a priori not related to the construction presented here. The invariance condition implies in particular that the functions χ, ψ depend on the $\mathbb{R}$-coordinate only, and are independent of coordinates of the base space.

The simplest ansatz that satisfies equation (9.7) in the Sasakian case is $X_1 = \chi(\tau)I_1, X_a = \psi(\tau)I_a$ with real functions $\chi(\tau), \psi(\tau)$, where the invariance condition is still satisfied after individual rescaling of X_1 or X_a by a real factor. With this ansatz, the connection takes the form

$$\mathcal{A} = e^i I_i + \chi(\tau)e^1 I_1 + \frac{1}{\sqrt{2m}}\psi(\tau)e^a I_a. \tag{9.8}$$

The second summand has been suitably rescaled in order to simplify the following computation. In the following, we will furthermore assume that our Sasakian manifold has coset structure. As an example, one can think of the sphere $S^{2m+1} =$

$SU(m+1)/SU(m)$. The following computation will therefore bear close analogy to the computation on Chapter 8. The coset structure enables us in particular to relate the one-forms $\{e^\mu, e^i\}$ as $e^\mu = e^i_\mu e^i$ via real functions e^i_μ.

9.1 Yang-Mills Equation with Torsion

Let us write out the torsionful Yang-Mills equation (8.3) in this setup. For specification of the torsion term in the Yang-Mills equation, we have to compute the components of the three-form $\mathcal{H}$ in such a way that the instanton case with $*\mathcal{H} = d * Q$ is recovered for $\kappa = 1$. According to [29], the G-structure four-form on the cylinder over a Sasakian manifold with metric (9.2) is given by

$$Q_{\mathcal{Z}} = \frac{2m}{m+1} d\tau \wedge P + \left(\frac{2m}{m+1}\right)^2 Q, \tag{9.9}$$

with P and Q as in equation (4.13). We take the instanton case as starting point. Using the definition $*\mathcal{H} := d * Q$, the fact that

$$* * \rho = (-1)^{r(d-r)} \rho \tag{9.10}$$

holds for an r-form ρ on a d-dimensional Riemannian manifold, as well as decomposition rules for antisymmetric tensor indices, a direct computation yields

$$\mathcal{H}_{ABC} = -\frac{5}{2} f^{MN}{}_{[C} Q_{MNAB]} = -\frac{15}{2} Q_{MN[AB} f^{MN}{}_{C]}, \tag{9.11}$$

where capital indices label directions on $\mathcal{Z}(M)$. The upper indices of $\mathcal{H}^A_{BC}$, f^A_{BC} and T^A_{BC} will always be lowered such that they appear *behind* the lower two ones, i. e. $g_{AA} f^A_{BC} = f_{BCA}$. This convention is important, as not all quantities can a priori be assumed to be totally antisymmetric. At this point, we have to distinguish between indices in cylinder direction (0), contact direction (1) and all other directions and find that the following components vanish in any dimension:

$$\mathcal{H}_{01c} = \mathcal{H}_{0bc} = \mathcal{H}_{abc} = 0. \tag{9.12}$$

The remaining components depend on the value of m. We demonstrate this by writing out $\mathcal{H}_{231}$ explicitly, using equations (4.13), (4.17) and (9.5). All other nonvanishing components of $\mathcal{H}$ behave in a similar way.

$$
\begin{aligned}
&\mathcal{H}_{231} = -2P_{231}Q_{2323} = 0 && \text{for } m = 1,\\
&\mathcal{H}_{231} = -\frac{9}{16}P_{mn1}Q_{mn23} = -2P_{451} = -f_{231} && \text{for } m = 2,\\
&\mathcal{H}_{231} = -\frac{4}{9}P_{mn1}Q_{mn23} = -2(P_{451} + P_{671}) = -2f_{231} && \text{for } m = 3,\\
&\mathcal{H}_{231} = (1-m)f_{231} && \text{for arbitrary } m.
\end{aligned}
\tag{9.13}
$$

Note that the case $m = 1$ of lowest dimension with $\mathcal{H} = 0$ is special. We will not further discuss it here. In order to recover the instanton case for $\kappa = 1$, we choose

$$
H_{\mu\nu\rho} = (1-m)T_{\mu\nu\rho} = (1-m)\kappa f_{\mu\nu\rho}. \tag{9.14}
$$

With this choice and the cylinder metric (9.2), equation (8.3) turns into

$$
\begin{aligned}
\partial_A\mathcal{F}^{AB} - \mathcal{F}^{CD}\left(\frac{1}{2}(2-m)T^B_{CD} - {}^-\Gamma^B_{CD}\right)&\\
+ \mathcal{F}^{CB}\left(\frac{1}{2}T^D_{CD} - {}^-\Gamma^D_{CD}\right) - \mathcal{F}^{CB}\left(\frac{1}{2}T^D_{DC} - {}^-\Gamma^D_{DC}\right) &+ [\mathcal{A}_A, \mathcal{F}^{AB}] = 0,
\end{aligned}
\tag{9.15}
$$

where ${}^-\Gamma$ denotes the torsionful spin connection as in Chapter 8. The $B = 0$ equation is identically satisfied. Let us take a look at the cases with $B \neq 0$. The summand $\mathcal{F}^{C\mu}(\frac{1}{2}T^D_{CD} - {}^-\Gamma^D_{CD})$ vanishes identically. From $\mathcal{F}^{C\mu}(\frac{1}{2}T^D_{DC} - {}^-\Gamma^D_{DC})$ and $[\mathcal{A}_A, \mathcal{F}^{AB}]$, we obtain terms proportional to the functions e^i_μ. These terms add up to zero by use of the Jacobi identity and condition (9.7) and will therefore be omitted in the following computation. We evaluate the remaining terms explicitly. With

$$
de^A = -\frac{1}{2}f^A_{BC}e^{BC} \qquad \text{and} \qquad T^A_{BC} = \kappa f^A_{BC}, \tag{9.16}
$$

the coefficients of the spin connection ${}^-\Gamma$ take the form

$$
{}^-\Gamma^1_{bc} = \frac{1}{2}(\kappa+1)f^1_{bc}, \tag{9.17}
$$

$$
{}^-\Gamma^a_{1b} = \frac{1}{2}(\kappa+1)f^a_{1b} + f^a_{ib}e^i_1, \tag{9.18}
$$

$$
{}^-\Gamma^a_{bc} = f^a_{ic}e^i_b, \tag{9.19}
$$

similar to (8.6) on G/H. Using equation (9.8) and omitting the τ-dependence of the functions χ and ψ, we obtain the following curvature:

$$\begin{aligned}\mathcal{F} = &-\frac{1}{2}\left(1-\frac{1}{2m}\psi^2\right) f^i_{ab}e^{ab}I_i + \dot{\chi}e^{01}I_1 + \frac{1}{\sqrt{2m}}\dot{\psi}e^{0a}I_a \\ &+\left(\chi-\frac{1}{2m}\psi^2\right) P_{ab1}e^{ab}I_1 + \frac{m+1}{m\sqrt{2m}}\psi(1-\chi)P_{1ba}I_a e^{1b}. \end{aligned} \tag{9.20}$$

Inserting $\mathcal{F}$, T^C_{AB}, ${}^-\Gamma^C_{AB}$ as above and using

$$f^i_{ac}f^c_{ib} = \frac{2(m^2-1)}{m}\delta_{ab}, \tag{9.21}$$

equation (9.15) turns into the second-order equations

$$\ddot{\chi} = \frac{(m+1)^2}{m}\left(\left((m-1)\kappa+1\right)\chi - \left((m-1)\kappa+3\right)\frac{1}{2m}\psi^2 + \frac{1}{m}\chi\psi^2\right), \tag{9.22a}$$

$$\ddot{\psi} = \left(\frac{m+1}{m}\right)^2 \psi\left((m-1)\kappa+2-m-\left((m-1)\kappa+3\right)\chi+\chi^2+\frac{1}{2}\psi^2\right). \tag{9.22b}$$

The proof of the identity (9.21) can be found in Appendix C.3.

9.2 Action Functional and Potential

The second-order equations (9.22) are equations of motion for the action

$$\begin{aligned} S &= \frac{m}{4(m+1)}\int_{\mathbb{R}\times M} \mathrm{tr}\left(\mathcal{F}\wedge *\mathcal{F} + 2\left(\frac{m}{m+1}\right)^2 \kappa d\tau\wedge *_M Q_M\wedge\mathcal{F}\wedge\mathcal{F}\right) \\ &= Vol(M)\times\int_{\mathbb{R}}\Bigg[-\frac{1}{2}(\dot{\chi}^2+\dot{\psi}^2) - \left(\frac{m+1}{m}\right)^2 \\ &\qquad \Bigg(\psi^2(1-\chi)^2 + m(1-m)(1-\kappa)\left(\frac{1}{2m}\psi^2-1\right)^2 \\ &\qquad +m(1+\kappa(m-1))\left(\chi-\frac{1}{2m}\psi^2\right)^2\Bigg)\Bigg]d\tau \end{aligned} \tag{9.23}$$

with potential

$$\begin{aligned} V(\chi,\psi) = &\frac{1}{2}\left(\frac{m+1}{m}\right)^2 \Big(((1+\kappa(m-1))m\chi^2 + \left(\kappa(1-m)-3\right)\chi\psi^2 \\ &+\chi^2\psi^2 + \left(2-m+\kappa(m-1)\right)\psi^2 + \frac{1}{4}\psi^4 + m(m-1)(1-\kappa)\Big), \end{aligned} \tag{9.24}$$

where $*_M$ denotes the Hodge star operator on the Sasakian manifold M with respect to the metric $g_M = e^1e^1 + \frac{2m}{m+1}\delta_{ab}e^ae^b$, $*$ denotes the Hodge star operator on the cylinder, and $Vol(M) = \sqrt{|g_M|}e^{12\cdots(2m+1)}$ is the volume form on M. An explicit computation of the action can be found in Appendix C.3. Equations (9.22) constitute a gradient system of the form

$$\begin{pmatrix} \ddot{\chi} \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} \partial_\chi \\ \partial_\psi \end{pmatrix} V. \tag{9.25}$$

The potential V is symmetric with respect to sign changes of ψ and has the following critical points (i. e. $\ddot{\chi} = \ddot{\psi} = 0$) for arbitrary m, κ:

$$\begin{aligned}
(\chi_1, \psi_1) =&(0, 0),\\
(\chi_2, \psi_2) =&(1, \pm\sqrt{2m}),\\
(\chi_3, \psi_3) =& \left(\frac{1}{4}\left(7 + 3(m-1)\kappa + \sqrt{P}\right),\right.\\
&\left.\pm\frac{1}{2}\sqrt{((1-m)\kappa - 1)\left((1-m)\kappa - 1 + 4m + \sqrt{P}\right)}\right),\\
(\chi_4, \psi_4) =& \left(\frac{1}{4}\left(7 + 3(m-1)\kappa + \sqrt{P}\right),\right.\\
&\left.\pm\frac{1}{2}\sqrt{((1-m)\kappa - 1)\left((1-m)\kappa - 1 + 4m - \sqrt{P}\right)}\right),
\end{aligned} \tag{9.26}$$

where the abbreviation

$$P = (m-1)^2\kappa^2 + \kappa(8m^2 - 6m - 2) + 24m + 1 \tag{9.27}$$

is used. Finite-action Yang-Mills solutions $\chi(\tau), \psi(\tau)$ must interpolate between zero-potential critical points. With κ arbitrary, the potential vanishes for the second critical point $(\chi_2, \psi_2) = \left(1, \pm\sqrt{2m}\right)$. We find $V(\chi_1, \psi_1) = \frac{(\kappa-1)(m-1)(m+1)^2}{2m}$ for the first critical point, which vanishes only for $\kappa = 1$, as well as lengthy nonzero expressions for $V(\chi_3, \psi_3)$ and $V(\chi_4, \psi_4)$. The critical points together with the κ-values for which their potential becomes zero are listed in Table 6. Table 7 lists the special values of κ for which more than two critical points are located on the same axis, and hence the system may admit analytic solutions. In addition, we note that at $\kappa = \frac{m-2}{m-1}$, five of the seven critical points coincide at $(0, 0)$, at $\kappa = \frac{m-2-\sqrt{m(8+m)}}{2(m-1)}$ the point (χ_3, ψ_3) coincides with (χ_2, ψ_2) and (χ_4, ψ_4) becomes imaginary, and at $\kappa = \frac{2-m-\sqrt{m(8+m)}}{2(m-1)}$, (χ_4, ψ_4) coincides with (χ_2, ψ_2) and (χ_3, ψ_3) becomes imaginary.

	κ	Eigenvalues of Hessian
$(\chi_1, \psi_1) = (0, 0)$	1	$(m+1)^2, \frac{(m+1)^2}{2}$
$(\chi_2, \psi_2) = (1, \pm\sqrt{2m})$	any	see Appendix C.3
$(\chi_3, \psi_3) = (1, -\sqrt{2m})$	$\frac{m-2-\sqrt{m(8+m)}}{2(m-1)}$	0, positive
$(\chi_4, \psi_4) = (-1, \pm\sqrt{2m})$	$\frac{3}{1-m}$	$\frac{(m+1)^2\left(m\mp\sqrt{m(m+8)}\right)}{m^2}$
$(1, \sqrt{2m})$	$\frac{m-2+\sqrt{m(8+m)}}{2(m-1)}$	0, positive

Table 6: Critical points and corresponding κ values with vanishing potential.

κ	Critical points	V(critical points)
$\dfrac{1}{1-m}$	$(0,0), (1, \pm\sqrt{2m}), (1 \pm m, 0)$	$\frac{1}{2}(m+1)^2, 0, \frac{1}{2}(m+1)^2$
$\dfrac{3}{1-m}$	$(0,0), (1, \pm\sqrt{2m}), (-1, \pm\sqrt{2m}),$ $(0, \pm\sqrt{2(m+1)})$	$\frac{(m+1)^2(m+2)}{2m}, 0, 0, -\frac{(m+1)^2}{2m^2}$

Table 7: Values of κ for which more than two critical points lie on the same axis.

9.3 Analytic Yang-Mills Solutions

The case $\kappa = \frac{1}{1-m}$ admits an analytic solution to the Yang-Mills equation, interpolating between the critical points $(1, \sqrt{2m})$ and $(1, -\sqrt{2m})$ for arbitrary m. All other critical points are located on the χ-axis and have potential $V = \frac{1}{2}(m+1)^2$. The zero-potential critical points are therefore minima of V, and we expect to find interpolating finite-action Yang-Mills solutions. With $\chi = 1$, equations (9.22) take the form

$$\ddot{\chi} = 0 \tag{9.28a}$$

$$\ddot{\psi} = \frac{(m+1)^2}{m}\psi\left(\frac{1}{2m}\psi^2 - 1\right). \tag{9.28b}$$

Equation (9.28) can be integrated to the first-order equation

$$\dot{\psi} = \pm\frac{m+1}{\sqrt{m}}\psi\sqrt{\frac{1}{4m}\psi^2 + 1}, \tag{9.29}$$

which is solved by

$$\psi = \pm\sqrt{2m}\tanh\left(\pm\frac{m+1}{\sqrt{2m}}\tau\right). \tag{9.30}$$

This is a kink solution with finite energy and finite action. A plot of this solution in the χ, ψ-plane can be found in Figure 6.

For $\kappa = \frac{3}{1-m}$, there are three critical points on the $\chi = 0$ axis. However, none of them has zero potential, and we do not find any analytic solutions.

9.4 Periodic Solutions

A different kind of solution is obtained by changing from $\mathbb{R} \times M$ to $S^1 \times M$, i. e. when the additional direction is not a real line but a circle with circumference L. In this case, periodic boundary conditions have to be imposed:

$$\psi(\tau) = \psi(\tau + L). \tag{9.31}$$

We restrict the consideration to the analytically solvable case (9.28), which has the periodic solution

$$\psi(\tau) = \pm\frac{2k\sqrt{m}}{\sqrt{1+k^2}}\,\mathrm{sn}\left[\frac{m+1}{\sqrt{m(1+k^2)}}\tau; k\right]. \tag{9.32}$$

This solution is known as a sphaleron [12]. $\mathrm{sn}[u,k]$ with $0 \leq k \leq 1$ is a Jacobi elliptic function, details of which can be found for example in Appendix B of [26] or in [85]. The Jacobi elliptic function has a period of $4K(k)$, where $K(k)$ denotes the complete elliptic integral of the first kind. The boundary condition (9.31) therefore turns into

$$4K(k)n = \frac{m+1}{\sqrt{m(1+k^2)}}L, \qquad\qquad n \in \mathbb{N}, \tag{9.33}$$

fixing $k = k(L,n)$ and $\psi(\tau; k(L,n)) =: \psi^{(n)}(\tau)$. Solutions (9.32) exist if $L \geq 2^{\frac{3}{2}}\pi n$ (cf. [26, 75, 86]). The topological charge of the sphaleron $\psi^{(n)}$ is zero due to the periodic boundary conditions. This solution is interpreted as a configuration of n kinks and n antikinks, alternating and equally spaced around the circle. The tanh-solution from Chapter 9.3 arises from the Jacobi elliptic function in the limit $k \to 1$. In the limit $k \to 0$, the elliptic function approaches $\sin\left(\frac{m+1}{\sqrt{m(1+k^2)}}\tau\right)$. In analogy to results in [86], our solution (9.31) with positive sign has the following total energy, with $E(k)$ denoting the complete elliptic integral of the second kind:

$$\begin{aligned} E[\psi] &= \int_0^L d\tau \left(\frac{1}{2}(\partial_\tau\psi)^2 + V(1,\psi)\right) \\ &= \frac{\sqrt{2}\cdot 4nm^2(m+1)}{3(1+k^2)^{\frac{3}{2}}} \\ &\quad \left(\frac{1}{4m^2}\left(3k^4 + (6+32m^2+24m)k^2 + 16m^2 - 24m + 3\right)K(k)\right. \\ &\quad \left. + 2\left(\frac{3}{m} - 2\right)(1+k^2)E(k)\right). \end{aligned} \tag{9.34}$$

9.5 Dyons

Replacing the coordinate τ in $\mathbb{R}$-direction by $i\tau$ changes the signature of the metric from Riemannian to Lorentzian:

$$g = -e^0e^0 + e^1e^1 + e^{2h}\delta_{ab}e^{ab}. \tag{9.35}$$

The Yang-Mills equations (9.22) remain unchanged, except for the fact that the second-order derivatives now come with a minus sign:

$$(\ddot{\chi}, \ddot{\psi}) \to (-\ddot{\chi}, -\ddot{\psi}). \tag{9.36}$$

This corresponds to a sign flip of the potential, so that we have to study V instead of $-V$. Dyons are finite-energy solutions to the second-order equations obtained by this sign flip. Just as Yang-Mills solutions, they can interpolate between two critical points (kink), or start and end at the same point (bounce). Solutions that oscillate around a minimum can exist as well, but they do not lead to finite energy and hence will not be considered in the following. There are no analytic dyon solutions in our case, as will be argued in the following section. The construction of numerical dyon solutions is possible, and results are presented in Figure 7.

9.6 Discussion and Summary

Recall that in our sign convention, instanton solutions interpolate between minima and dyon solutions between maxima of V. In both cases, solutions that start or end at a saddle point are possible as well. With this in mind, we can list the expected solutions. We include special k-values of the case $m = 2$ in our list, as plots of the potential and numerical solutions will be presented for this case in Figures 4 to 7.

- κ arbitrary: there exist at least two zero-potential critical points at $(0, \pm\sqrt{2m})$ for all κ. According to Appendix C.3, they can be minima or saddle points of V, depending on the value of κ. This means that we can always find interpolating solutions, either of dyon or Yang-Mills type. These solutions have to be constructed numerically unless $\kappa = \frac{1}{1-m}$.

- $\kappa = 1$: this is the instanton case. Yang-Mills solutions exist between $(0, 0)$ and $(1, \pm\sqrt{2m})$ (cf. [29]). We do not expect to find finite-action dyon solutions, as the zero-potential critical points of V are minima.

- $\kappa = \frac{1}{1-m}$ ($\kappa = -1$ for $m = 2$): in this case, we find three critical points with nonzero potential along the χ axis. An analytic Yang-Mills solution interpolates between the two remaining zero-potential critical points, which are minima for all m. This solution for arbitrary m has been presented in Chapter 9.3.

- $\kappa = \frac{3}{1-m}$ ($\kappa = -3$ for $m = 2$): we find four zero-potential critical points. Two of them are located at the lines with $\chi = 1$ and $\chi = -1$, respectively. We do not find any analytic solutions along the $\chi = \pm 1$ lines and $\chi = 0$

	κ	Eigenvalues of Hessian
$(\chi_1, \psi_1) = (0, 0)$	1	$9, \frac{9}{4}$
$(\chi_2, \psi_2) = (1, \pm 2)$	any	$\frac{9}{4}\left(5 + \kappa \pm \sqrt{5}(1+\kappa)\right)$
$(\chi_3, \psi_3) = (1, -2)$	$-\sqrt{5}$	$\frac{9}{2}\left(5 - \sqrt{5}\right), 0$
$(\chi_4, \psi_4) = (-1, \pm 2)$	-3	$\frac{9}{2}\left(1 \pm \sqrt{5}\right)$
$(1, 2)$	$\sqrt{5}$	$\frac{9}{2}\left(5 + \sqrt{5}\right), 0$

Table 8: Critical points and corresponding κ-values with vanishing potential for $m = 2$.

axes. There are, however, numerical solutions interpolating between various pairs of critical points.

We do not expect any analytic dyon solutions, as the zero-potential critical points are minima in the analytically solvable cases. For a better understanding, we present the case $m = 2$ as an example. The potential for various interesting values of κ is shown in Figure 4, and further dyon and Yang-Mills solutions for this example are presented in Figures[24] 5 and 7. A list of zero-potential critical points for this case can be found in Table 8.

[24] A Mathematica code for the creation of the presented plots can be found in [76].

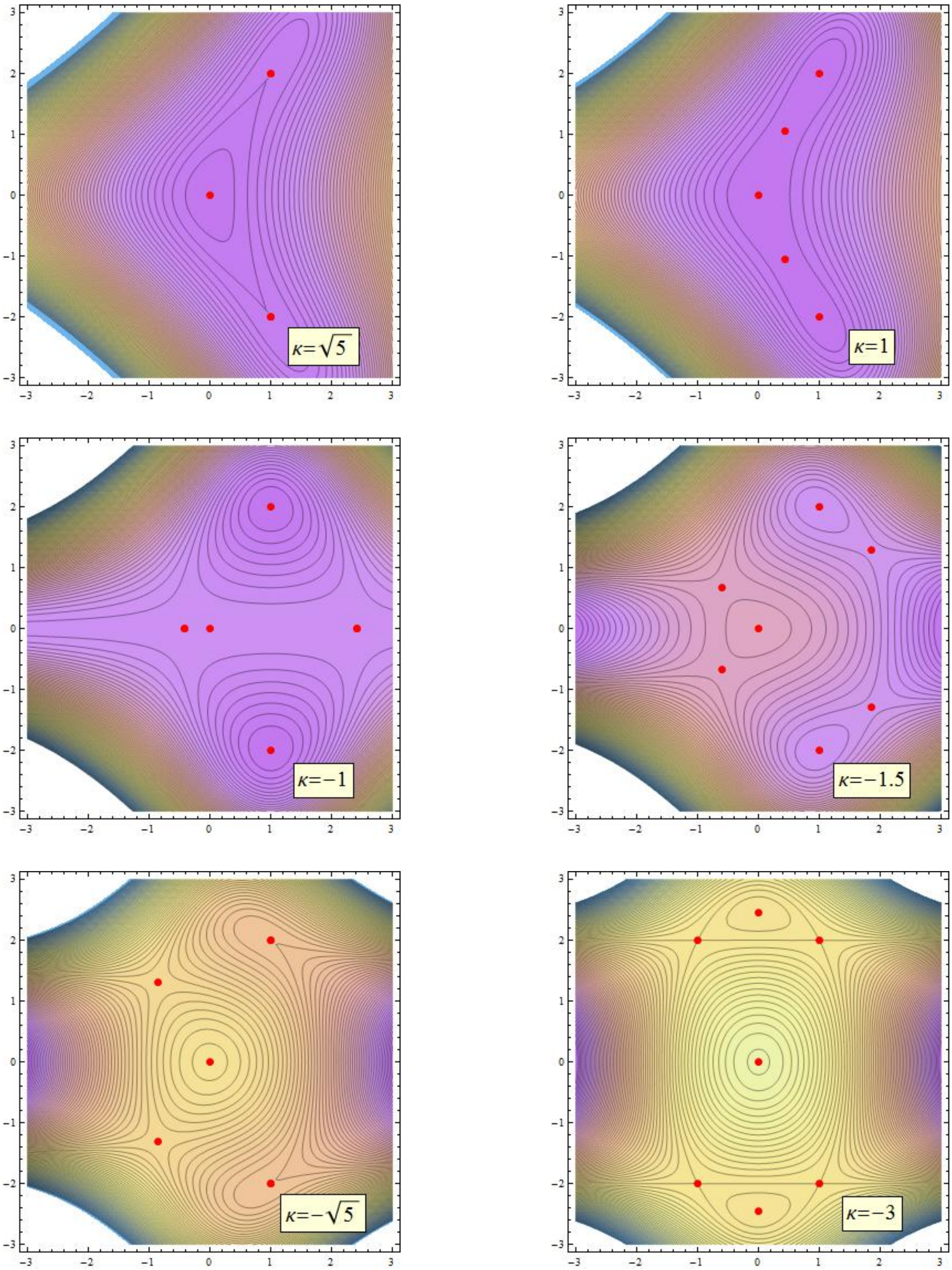

Figure 4: Plots of the negative of the potential (9.24) for various values of κ and $m = 2$.

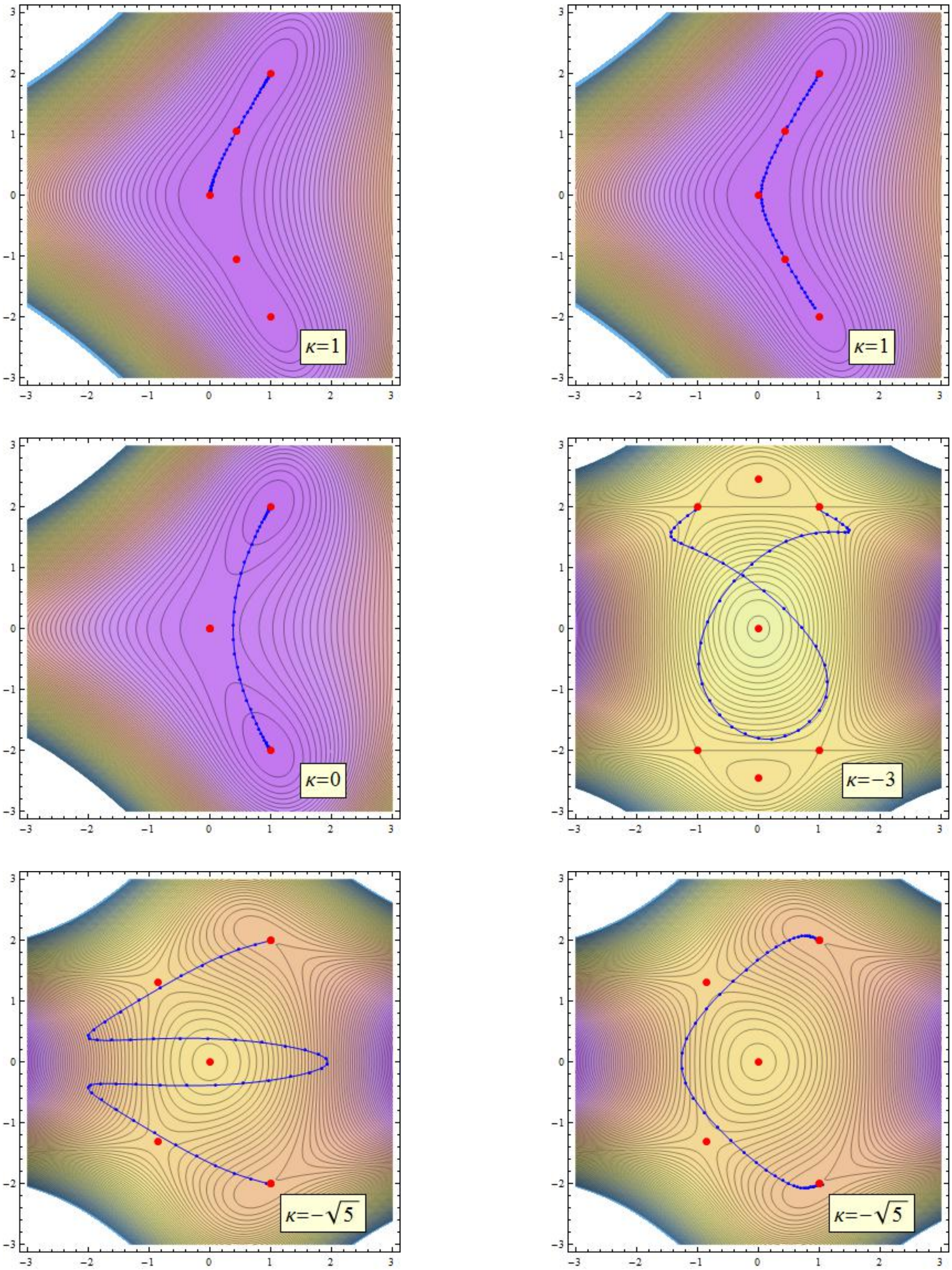

Figure 5: Some solutions of the Yang-Mills equation for various values of κ and $m = 2$.

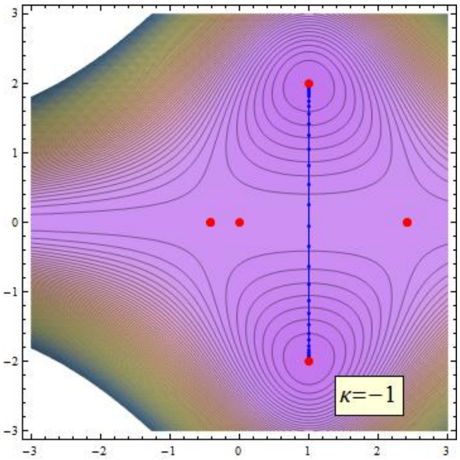

Figure 6: Analytic solution of the Yang-Mills equation (9.28) for $\kappa{=}-1$ and $m{=}2$.

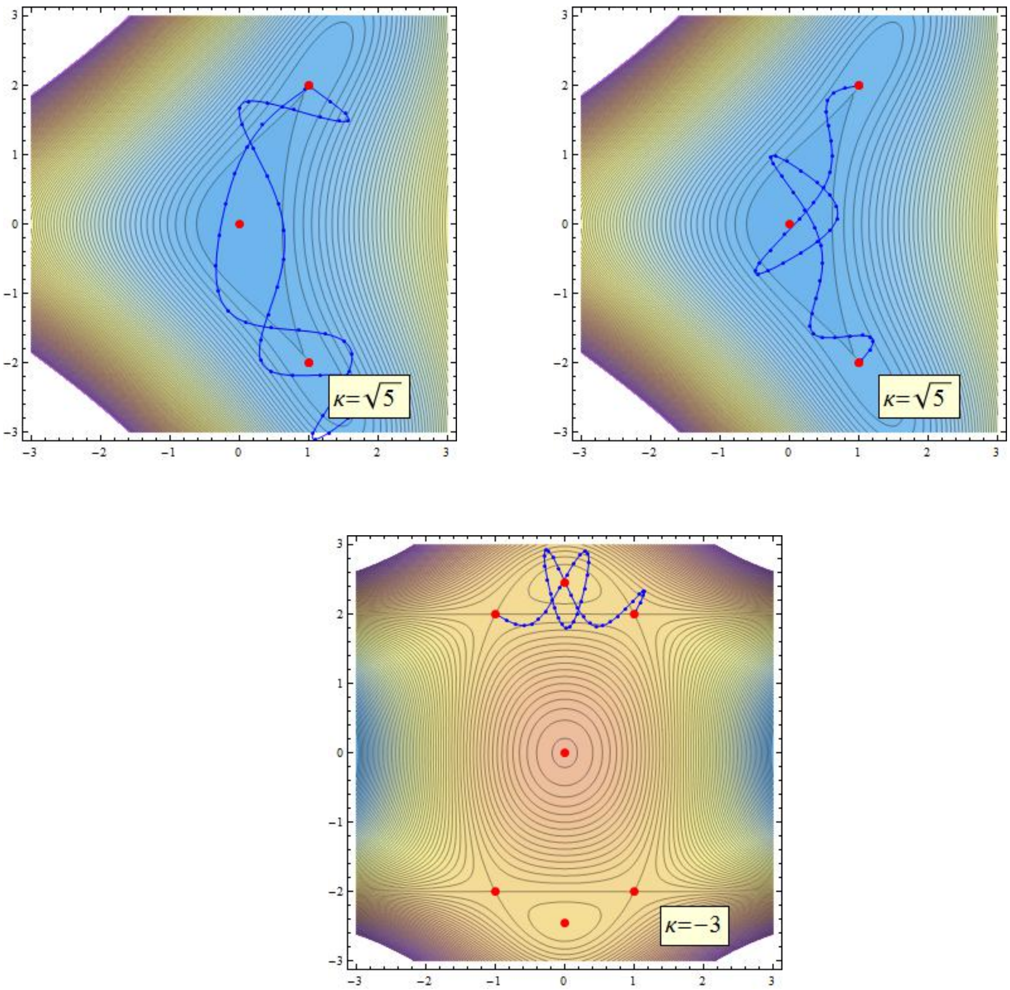

Figure 7: Some numerical dyon solutions for various values of κ and $m = 2$.

10 Conclusion and Outlook

Instantons

In Chapters 6 and 7, we have studied solutions of the instanton equation on the cylinder and were able to reproduce known kink-type solutions with the simplest ansatz for a gauge connection $\mathcal{A}$ with one scalar function. A generalization of $\mathcal{A}$ did not allow for the construction of explicit solutions. In the most general case, the instanton equation leads to algebraic relations which can be interpreted as quiver relations (cf. [73]). It can be shown that our equations (6.47) and (6.48) match the $SU(3)$-quiver relations presented in [73]. A more detailed comparison to quiver relations in the case of general G/H may lead to a better understanding of our algebraic conditions.

Turning to explicit examples, we decided to investigate the cylinder over a restricted half-flat coset space. We have presented two G_2-structure four-forms and the corresponding instanton equations on $\mathbb{R} \times SU(3)/(U(1) \times U(1))$, leaving us with a couple of questions. The instanton conditions derived with the first four-form (7.5) seem rather restrictive, but may allow for the construction of solutions with three individual functions ϕ_1, ϕ_2, ϕ_3 and differing deformation parameters R_1, R_2, R_3. This is an interesting question for future work, as explicit instantons on this space have only been constructed in the nearly-Kähler case, or found to restrict the geometry to being nearly-Kähler. We may also ask whether there are more general closed four-forms that can serve for the construction of instantons. In this context, the study of cohomology of forms on the chosen space may lead to new insights.

As in the search for non-BPS Yang-Mills solutions, we can extend the study of instantons to more general geometries and other dimensions. In six dimensions, the half-flat space $SU(2)\times SU(2)$ appears interesting, as it admits a number of deformation parameters [49] and has only been considered before with nearly-Kähler structure in the context of instantons [87]. The study of instantons on cylinders over seven-dimensional coset spaces (for example the squashed seven-sphere) is work in progress. We expect similar results as in the case of six-dimensional $SU(3)$-structure base manifolds.

One would obtain an overview of possible instanton solutions and their parametrization if more information about the instanton moduli space was available. This is an interesting task for future work, but it appears rather hard to access, as the dimension of the considered spaces is not a multiple of four, and known methods such as the ADHM construction cannot be applied straight away.

Non-BPS Yang-Mills Solutions

Using a special ansatz for the gauge connection, we have derived an explicit second-order Yang-Mills equation on the cone over a general coset space G/H in Chapter 8. As already mentioned, the next step for the interpretation of the constructed solutions is to determine the constant term of the potential by an explicit computation of the action. We expect the requirement of finite action to restrict the possible values of the parameter κ and lead to a condition on the opening angle γ of the cone.

In Chapter 9, we have derived a system of explicit second-order Yang-Mills equations on the cylinder over a class of Sasakian manifolds. We have constructed the corresponding action and potential, discussed the behaviour of the critical zero-potential points and found analytic as well as numerical solutions of Yang-Mills, dyon and sphaleron type.

A similar discussion for cylinders over certain $SU(3)$-structure manifolds can be found in [26]. A comparison with our results illustrates that Sasakian and $SU(3)$-structures are fundamentally different. The perhaps most striking fact is that the 3-symmetry of the $SU(3)$-structure manifold is recovered in the shape of the potential, whereas the potential in the Sasakian case is symmetric only under sign changes of the variable ψ. Furthermore, the Sasakian potential does not admit as many solutions with straight trajectories in the (χ, ψ)-plane as the $SU(3)$-structure potential does. In the latter case, the distribution of κ-dependent and κ-independent zero-potential critical points allows to systematically associate certain types of solutions (kinks, bounces) to intervals of the deformation parameter κ. In particular, there are always three critical points on the real axis. The Sasakian potential admits fewer κ-independent zero-potential critical points, and they are not as regularly distributed as in the $SU(3)$-structure case. The range and type of our solutions is therefore significantly different.

In spite of these differences, we have found that Sasakian manifolds do admit various interesting solutions. This, and in particular the construction of an analytic kink-type Yang-Mills solution, makes them potentially interesting for string compactifications. We leave the interpretation of the first-order solution (9.30) in this context for future work.

To complete the discussion, it would be interesting to apply our method to cylinders over G_2-structure manifolds, i. e. eight-dimensional manifolds with $Spin(7)$-structure, as well as cylinders over 3-Sasakian manifolds, which have a structure similar to the Sasakian ones studied here.

We have restricted the discussion in Chapter 9 to the Yang-Mills equation on the cylinder. The same analysis can be performed on the cone after adapting the four-form Q and using the Yang-Mills equation (8.5). As we have seen in Chapter 8, the second-order equations on conical manifolds acquire a first-order friction term, hence the explicit analysis might have to be done numerically. Finally, the study of Yang-Mills solutions on cones and cylinders may be extended to sine-cones (cf. [28]).

After concentrating on the technical details of the construction of dyon and sphaleron solutions, another interesting point is their interpretation in the framework of particle physics, including global symmetries and charges.

Appendix

A Manifolds with Almost Complex Structure

Let us review the most important facts about complex and almost complex structures that are used for the introduction of manifolds with $SU(3)$- and Sasakian structure. For a more complete review, we refer to [47, 67, 88] and [89].

Let M denote a real $2n$-dimensional manifold. M is called **almost complex** if it admits a globally defined almost complex structure, that is a rank $(1,1)$-tensor J satisfying

$$J^2 = -\mathbb{1}. \tag{A.1}$$

Let $\{E_A\}$, $A = (1, \ldots, 2n)$, be a basis of real vector fields on the tangent bundle TM and $\{e^A\}$ the dual basis of one-forms. Being a rank $(1,1)$-tensor, J can locally be written in this real basis as

$$J = J_A^B e^A \otimes E_B. \tag{A.2}$$

The components satisfy

$$J_A^C J_C^B = -\delta_A^B, \tag{A.3}$$

according to equation (A.1). It can be shown that almost complex manifolds are always of even dimension.

The existence of an almost complex structure allows to define the notion of **holomorphicity degree** of differential forms, refining the structure of r-forms to the structure of (p,q)-forms. At every point $p \in M$, the tensor J acts on the complexified tangent space $T_pM_{\mathbb{C}}$ as a map that squares to $-\mathbb{1}$ and has eigenvalues $\pm i$. We denote the eigenspaces of J as $T_pM_{\mathbb{C}}^{(1,0)}$ and $T_pM_{\mathbb{C}}^{(0,1)}$ and their elements as holomorphic and antiholomorphic vectors. Elements of the corresponding cotangent spaces $T_p^*M_{\mathbb{C}}^{(1,0)}$ and $T_p^*M_{\mathbb{C}}^{(0,1)}$ are called holomorphic and antiholomorphic one-forms. The complexified tangent and cotangent bundles split accordingly, and we refer to $T^*M_{\mathbb{C}}^{(1,0)}$ and $T^*M_{\mathbb{C}}^{(0,1)}$ as holomorphic and antiholomorphic cotangent bundle. Taking exterior powers of these bundles, we find

$$\wedge^{(p,q)}T^*M_{\mathbb{C}} := \left(\wedge^p(T^*M_{\mathbb{C}}^{(1,0)})\right) \wedge \left(\wedge^q(T^*M_{\mathbb{C}}^{(0,1)})\right), \tag{A.4}$$

allowing us to define the space of (p,q)-forms on the almost complex manifold M as

$$\Omega^{(p,q)}(M) := \Gamma(\wedge^{(p,q)}T^*M_{\mathbb{C}}). \tag{A.5}$$

The space of r-forms decomposes into a sum of (p,q)-form spaces as

$$\Omega^r(M) = \bigoplus_{p+q=r} \Omega^{(p,q)}(M). \tag{A.6}$$

When $p+q=2$, for example, a 2-form $\omega \in \Omega^2(M)$ splits according to

$$\omega = \omega^{(2,0)} + \omega^{(1,1)} + \omega^{(0,2)}. \tag{A.7}$$

We can introduce a local basis of n holomorphic one-forms with index $\alpha = (1,\ldots,n)$ and n anti-holomorphic one-forms with index $\bar{\alpha} = (\bar{1},\ldots,\bar{n})$ of the complexified cotangent bundle $T^*M_{\mathbb{C}}$ as linear combinations of the above real, non-holonomic one-forms $\{e^A\}$:

$$\xi^\alpha = \frac{1}{2}(e^\alpha + ie^{\alpha+n}), \qquad \bar{\xi}^{\bar{\alpha}} = \frac{1}{2}(e^{\bar{\alpha}} - ie^{\bar{\alpha}+n}). \tag{A.8}$$

In this complex basis, a (p,q)-form ω is written as a linear combination of p holomorphic and q antiholomorphic one-forms:

$$\omega = \frac{1}{p!q!}\omega_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}\xi^{\alpha_1\cdots\alpha_p} \wedge \bar{\xi}^{\bar{\beta}_1\cdots\bar{\beta}_q}. \tag{A.9}$$

A special case of almost complex manifolds are **complex manifolds**, which are defined as follows: Let M be a real manifold of even dimension. A complex chart on M is a pair (U,φ) of an open set $U \subset M$ and a diffeomorphism $\varphi : U \to \mathbb{C}^n$. M is said to be a **complex manifold** if it admits an atlas $\{(U_\alpha,\varphi_\alpha)\}$ of complex charts such that the transition functions

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \tag{A.10}$$

are biholomorphic.

This notion is related to that of an almost complex manifold via the concept of integrability. An almost complex structure J is called **integrable** if the Lie bracket of any two holomorphic vector fields is again a holomorphic vector field. It has been shown that an almost complex manifold (M, J) is complex if and only if J is integrable. An integrable almost complex structure is therefore called **complex structure**.

For any two vector fields $X, Y \in \Gamma(TM)$, we define the **Nijenhuis tensor** as

$$N(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]. \tag{A.11}$$

It can be shown that an almost complex structure is integrable if and only if the Nijenhuis tensor vanishes [90].

A special class of complex manifolds are **Kähler manifolds**. Before introducing them, we have to define the notion of hermiticity of the metric. Let M be a complex manifold with Riemannian metric g and complex structure J. The metric is said to be **Hermitean** if for any two vector fields $X, Y \in \Gamma(TM)$ the relation

$$g(JX,JY) = g(X,Y) \tag{A.12}$$

is satisfied. The manifold (M, g) is then referred to as Hermitean manifold.

In local real coordinates, the hermiticity condition on the metric reads

$$g_{AB} = J_A^M J_B^N g_{MN}. \tag{A.13}$$

In local holomorphic coordinates, we can write a Hermitean metric as

$$g = g_{\alpha\bar{\beta}} \xi^\alpha \otimes \bar{\xi}^{\bar{\beta}} + g_{\bar{\alpha}\beta} \bar{\xi}^{\bar{\alpha}} \otimes \xi^\beta. \tag{A.14}$$

We can introduce a **fundamental two-form** ω, also referred to as **Kähler form**, on a Hermitean manifold as

$$\omega(X,Y) := g(JX,Y), \tag{A.15}$$

where $X, Y \in \Gamma(TM)$. In local real coordinates, this form becomes

$$\omega_{AB} = J_A^C g_{CB}, \tag{A.16}$$

and the components satisfy $\omega_{AB} = -\omega_{BA}$. In local complex coordinates, we find

$$\omega = 2i g_{\alpha\bar{\beta}} \xi^\alpha \wedge \bar{\xi}^{\bar{\beta}}. \tag{A.17}$$

This implies that the fundamental two-form is of holomorphicity degree $(1, 1)$. A manifold (M, J) with complex structure is said to be **Kähler** if the fundamental form is closed. If (M, J) is almost complex and $d\omega = 0$, the manifold is called **almost Kähler**.

B Hodge Star Operator and the Levi-Civita Tensor

In this section, we collect some identities and definitions that are used for computations throughout the thesis, in particular for the derivation of the instanton and Yang-Mills equation in Chapter 6 and Appendix C.

Levi-Civita Tensor

For simplicity, let us restrict the consideration to manifolds with Euclidean signature. The presented identities can easily be generalized to manifolds with arbitrary signature, but this will not be needed for our work. Our conventions are adopted from [67]. On flat d-dimensional Euclidean space, we can define the **Levi-Civita symbol** as follows:

$$\varepsilon_{A_1 A_2 \cdots A_d} := \begin{cases} +1, & \text{if } (A_1 A_2 \cdots A_d) \text{ is an even permutation of } 1, 2, \ldots, d \\ -1, & \text{if } (A_1 A_2 \cdots A_d) \text{ is an odd permutation of } 1, 2, \ldots, d \\ 0, & \text{otherwise.} \end{cases} \tag{B.1}$$

Indices of this tensor are raised and lowered with δ_{AB}, implying

$$\varepsilon_{A_1 A_2 \cdots A_d} = \varepsilon^{A_1 A_2 \cdots A_d}. \tag{B.2}$$

The analogue of the Levi-Civita symbol on a curved space (M, g) is the totally antisymmetric tensor density ϵ, which is related to the flat Levi-Civita symbol as

$$\epsilon_{A_1 A_2 \cdots A_d} = \sqrt{|g|}\varepsilon_{A_1 A_2 \cdots A_d}, \tag{B.3}$$

with $|g| := \det(g)$ denoting the determinant of the metric. Indices of the **curved Levi-Civita tensor** are raised and lowered with g, implying

$$\epsilon^{A_1 A_2 \cdots A_d} = \frac{1}{\sqrt{|g|}} \varepsilon^{A_1 A_2 \cdots A_d} = \frac{1}{\sqrt{|g|}} \varepsilon_{A_1 A_2 \cdots A_d}. \tag{B.4}$$

The flat ε-symbol satisfies

$$\varepsilon^{A_1 \cdots A_p B_{p+1} \cdots B_d} \varepsilon_{A_1 \cdots A_p C_{p+1} \cdots C_d} = d!(d-p)! \delta^{B_{p+1} \cdots B_d}_{C_{p+1} \cdots C_d}, \tag{B.5}$$

where the Kronecker symbol with more than two indices is defined as

$$\delta^{B_{p+1} \cdots B_d}_{C_{p+1} \cdots C_d} := \delta^{B_{p+1}}_{[C_{p+1}} \cdots \delta^{B_d}_{C_d]}, \tag{B.6}$$

with the bracket denoting antisymmetrization of the lower indices. The curved ϵ-tensor satisfies a similar relation:

$$\epsilon^{A_1 \cdots A_p B_{p+1} \cdots B_d} \epsilon_{A_1 \cdots A_p C_{p+1} \cdots C_d} = d!(d-p)! \delta^{B_{p+1} \cdots B_d}_{C_{p+1} \cdots C_d}. \tag{B.7}$$

The determinant of the metric g with components g_{AB} can be written by use of the flat Levi-Civita symbol as follows:

$$|g| = \frac{1}{d!} \varepsilon^{A_1 \cdots A_d} \varepsilon^{B_1 \cdots B_d} g_{A_1 B_1} \cdots g_{A_d B_d}, \tag{B.8}$$

where each index is summed over and runs from 1 to d.

Hodge Star Operator

Let

$$\omega = \frac{1}{r!} \omega_{A_1 \cdots A_r} e^{A_1 \cdots A_r} \in \Omega^r(M) \tag{B.9}$$

be an r-form on a d-dimensional Riemannian manifold (M, g). The **Hodge star operator** $* : \Omega^r(M) \to \Omega^{d-r}(M)$ is a map from r-forms to $(d-r)$-forms, defined as

$$*\omega = \frac{1}{(d-r)!r!} \omega^{A_1 \cdots A_r} \epsilon_{A_1 \cdots A_r B_{r+1} \cdots B_d} e^{B_{r+1} \cdots B_d}. \tag{B.10}$$

Using equation (B.7), it can be shown that

$$* * \omega = (-1)^{r(d-r)} \omega. \tag{B.11}$$

Let us consider the cone $\mathcal{C}(M)$ with metric $g_{\mathcal{C}} = dr^2 + r^2 g_M$ over an n-dimensional manifold (M, g_M). We denote by $*_M$ the Hodge star operator on the base manifold. The Hodge star operator $*_{\mathcal{C}(M)}$ on the cone acts as follows on p-forms $\widetilde{\omega}_p \in \Omega^p(M)$ that have components only on the base manifold M:

$$*_{\mathcal{C}(M)}\widetilde{\omega}_p = r^{n-2p}\left(*_M\widetilde{\omega}_p\right) \wedge dr, \tag{B.12}$$

$$*_{\mathcal{C}(M)}\left(dr \wedge \widetilde{\omega}_p\right) = r^{n-2p} *_M \widetilde{\omega}_p. \tag{B.13}$$

We can generalize the Hodge star operator to a map $* : \Omega^r(M, V) \to \Omega^{d-r}(M, V)$ on differential forms with values in some vector space V. Given a basis $\{E_1, \ldots, E_d\}$ of the vector space V, a form $\eta \in \Omega^r(M, V)$ can be written as

$$\eta = \eta^A \otimes E_A \tag{B.14}$$

with $\eta^A \in \Omega^r(M)$. The Hodge star operator then acts as

$$*\eta := *(\eta^A) \otimes E_A \tag{B.15}$$

on the vector-valued form.

C Detailed Computations

C.1 Structure Constants on Coset Spaces

Let G/H be a homogeneous space, denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to the groups G and H and let $\{I_{\tilde{a}}\} = \{I_a, I_i\}$ be a basis of $\mathfrak{g}$-generators as introduced in Chapter 2. The structure constants with respect to this basis are denoted $f_{\tilde{a}\tilde{b}}^{\tilde{c}}$. We use indices $\tilde{a} = (1, \ldots, \dim G)$ to label directions on G. In the following, we collect some properties of the structure constants and prove the identities that are needed for the computations in Chapter 6.

The generators of the group G satisfy the **Jacobi identity**

$$\sum_{a,b,c\ cycl.} [[I_{\tilde{a}}, I_{\tilde{b}}], I_{\tilde{c}}] = 0, \tag{C.1}$$

which implies the following condition on the structure constants:

$$f_{[\tilde{a}\tilde{b}}^{\tilde{d}} f_{\tilde{c}]\tilde{d}}^{\tilde{e}} = 0, \tag{C.2}$$

where the brackets denote antisymmetrization of indices. Furthermore, recall that an antisymmetric tensor of rank $(n, 0)$ can be decomposed as

$$T_{[\tilde{a}_1 \cdots \tilde{a}_n]} = \frac{1}{n}\left(T_{\tilde{a}_1[\tilde{a}_2\cdots\tilde{a}_n]} - T_{[\tilde{a}_2|\tilde{a}_1|\tilde{a}_3\cdots\tilde{a}_n]} + T_{[\tilde{a}_2\tilde{a}_3|\tilde{a}_1|\tilde{a}_4\cdots\tilde{a}_n]} \pm \ldots \pm T_{[\tilde{a}_2\cdots\tilde{a}_n]\tilde{a}_1}\right), \tag{C.3}$$

with $|\tilde{a}|$ denoting indices that are excluded from antisymmetrization. In particular, we find

$$T_{[\tilde{a}\tilde{b}\tilde{c}\tilde{d}]} = \frac{1}{4}\left(T_{[\tilde{a}\tilde{b}\tilde{c}]\tilde{d}} - T_{[\tilde{a}\tilde{b}|\tilde{d}|\tilde{c}]} + T_{[\tilde{a}|\tilde{d}|\tilde{b}\tilde{c}]} - T_{\tilde{d}[\tilde{a}\tilde{b}\tilde{c}]}\right) \tag{C.4}$$

for a rank $(4, 0)$ tensor.

Let g_K denote the metric on G induced by the Killing form as in equation (2.18). With this metric, the structure constants with all indices lowered are totally antisymmetric. This can be seen by use of the Jacobi identity (C.2) and the explicit form of the Killing metric, $(g_K)_{\tilde{a}\tilde{b}} = f^{\tilde{d}}_{\tilde{a}\tilde{c}} f^{\tilde{c}}_{\tilde{d}\tilde{b}}$, as follows:

$$\begin{aligned} f_{\tilde{a}\tilde{b}\tilde{c}} &= f^{\tilde{d}}_{\tilde{a}\tilde{b}} g_{\tilde{d}\tilde{c}} \\ &= f^{\tilde{d}}_{\tilde{a}\tilde{b}} f^{\tilde{n}}_{\tilde{d}\tilde{m}} f^{\tilde{m}}_{\tilde{n}\tilde{c}} \\ &= \left(f^{\tilde{d}}_{\tilde{b}\tilde{m}} f^{\tilde{n}}_{\tilde{d}\tilde{a}} + f^{\tilde{d}}_{\tilde{m}\tilde{a}} f^{\tilde{n}}_{\tilde{d}\tilde{b}} \right) f^{\tilde{m}}_{\tilde{c}\tilde{n}} \\ &= -f^{\tilde{d}}_{\tilde{b}\tilde{m}} f^{\tilde{n}}_{\tilde{a}\tilde{d}} f^{\tilde{m}}_{\tilde{c}\tilde{n}} - f^{\tilde{d}}_{\tilde{m}\tilde{a}} f^{\tilde{n}}_{\tilde{d}\tilde{b}} f^{\tilde{m}}_{\tilde{n}\tilde{c}}. \end{aligned} \tag{C.5}$$

The expression in the last line is totally antisymmetric in $\tilde{a}, \tilde{b}$ and $\tilde{c}$, implying $f_{\tilde{a}\tilde{b}\tilde{c}} = f_{[\tilde{a}\tilde{b}\tilde{c}]}$.

We may therefore assume that the $\mathfrak{g}$-generators are normalized such that the structure constants with indices in the position $f^{\tilde{c}}_{\tilde{a}\tilde{b}}$ are cyclic. With this assumption, we find

$$\begin{aligned} f^{\tilde{e}}_{[\tilde{a}\tilde{b}} f^{\tilde{e}}_{\tilde{c}\tilde{d}]} &= \frac{1}{4} \left(f^{\tilde{e}}_{[\tilde{a}\tilde{b}} f^{\tilde{e}}_{\tilde{c}]\tilde{d}} + f^{\tilde{e}}_{[\tilde{a}\tilde{b}} f^{\tilde{e}}_{\tilde{c}]\tilde{d}} - f^{\tilde{e}}_{\tilde{d}[\tilde{a}} f^{\tilde{e}}_{\tilde{b}\tilde{c}]} - f^{\tilde{e}}_{\tilde{d}[\tilde{a}} f^{\tilde{e}}_{\tilde{b}\tilde{c}]} \right) \\ &= \frac{1}{2} \left(f^{\tilde{e}}_{[\tilde{a}\tilde{b}} f^{\tilde{e}}_{\tilde{c}]\tilde{d}} - f^{\tilde{e}}_{\tilde{d}[\tilde{a}} f^{\tilde{e}}_{\tilde{b}\tilde{c}]} \right) \\ &= f^{\tilde{e}}_{[\tilde{a}\tilde{b}} f^{\tilde{e}}_{\tilde{c}]\tilde{d}} \\ &= \frac{1}{3} \left(2 f^{\tilde{e}}_{\tilde{a}[\tilde{b}} f^{\tilde{e}}_{\tilde{c}]\tilde{d}} + f^{\tilde{e}}_{[\tilde{b}\tilde{c}]} f^{\tilde{e}}_{\tilde{a}\tilde{d}} \right). \end{aligned} \tag{C.6}$$

This holds in particular for structure constants on G/H and therefore implies equation (6.23). We use this and the following two identities for the derivation of equation (6.24):

$$\begin{aligned} f_{eca} f^e_{bd} f^i_{cd} &= \frac{1}{2} \left(f_{eca} f^e_{bd} f^i_{cd} + f_{eda} f^e_{bc} f^i_{dc} \right) \\ &= \frac{1}{2} \left(f_{eca} f^e_{bd} f^i_{cd} - f_{ecb} f^e_{ad} f^i_{cd} \right) \\ &= f_{ec[a} f^e_{b]d} f^i_{cd}, \end{aligned} \tag{C.7}$$

$$\begin{aligned} f_{eca} f^e_{bd} f^i_{cd} &= -f_{eca} (f^d_{bc} f^i_{ed} + f^d_{ce} f^i_{bd}) \\ &= -f_{cea} f^d_{be} f^i_{cd} - f_{eca} f^d_{ce} f^i_{bd} \\ &= -f_{eca} f^e_{bd} f^i_{cd} - \alpha f^i_{ab}. \end{aligned} \tag{C.8}$$

In this computation, we have used the Jacobi identity of the form $f^e_{[ab} f^e_{c]i} = 0$ and the fact that summation indices can be renamed. Combining the identities (C.7) and (C.8) yields equation (6.24).

In order to prove equation (6.26), we use the adjoint representation

$$\mathrm{ad} \;:\; \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \tag{C.9}$$

of the Lie algebra $\mathfrak{g}$ to simplify index notation. In this representation, the generators $I_{\widetilde{a}}$ take the form

$$(\mathrm{ad}\,(I_{\widetilde{a}}))^{\widetilde{c}}_{\widetilde{b}} = (I_{\widetilde{a}})^{\widetilde{c}}_{\widetilde{b}} = -f^{\widetilde{c}}_{\widetilde{a}\widetilde{b}} \in \mathfrak{gl}(\mathfrak{g}). \tag{C.10}$$

We now assume that $\mathfrak{g}$ splits as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ with the same notation as in Chapter 2. Due to the reductive splitting, the trace of any matrix $U \in \mathfrak{gl}(\mathfrak{g})$ decomposes as

$$\mathrm{tr}_{\,\mathfrak{g}}\, U = U_{\widetilde{x}\widetilde{x}} = U_{aa} + U_{ii} = \mathrm{tr}_{\,\mathfrak{m}}\, U + \mathrm{tr}_{\,\mathfrak{h}}\, U. \tag{C.11}$$

Under the assumption that the coset is naturally-reductive, i. e. $f^{j}_{ia} = 0$, we compute

$$\begin{aligned}
\mathrm{tr}_{\,\mathfrak{g}}(I_a I_b) &= f^{\widetilde{y}}_{a\widetilde{x}} f^{\widetilde{x}}_{b\widetilde{y}} \\
&= \underbrace{f^{y}_{ax} f^{x}_{by} + f^{i}_{ax} f^{x}_{bi}}_{\mathrm{tr}\,\mathfrak{m}} + \underbrace{f^{y}_{ai} f^{i}_{by}}_{\mathrm{tr}\,\mathfrak{h}} \\
&= -(\underbrace{\frac{1}{2}(1+\alpha)}_{-\,\mathrm{tr}\,\mathfrak{m}} + \underbrace{\frac{1}{2}(1-\alpha)}_{-\,\mathrm{tr}\,\mathfrak{h}})\delta_{ab} = -\delta_{ab},
\end{aligned} \tag{C.12}$$

$$\mathrm{tr}_{\,\mathfrak{g}}(I_i I_c) = f^{\widetilde{y}}_{i\widetilde{x}} f^{\widetilde{x}}_{c\widetilde{y}} = \underbrace{f^{\widetilde{y}}_{ij} f^{j}_{c\widetilde{y}}}_{\mathrm{tr}\,\mathfrak{h}} + \underbrace{f^{\widetilde{y}}_{ix} f^{x}_{c\widetilde{y}}}_{\mathrm{tr}\,\mathfrak{m}} = 0, \tag{C.13}$$

$$\begin{aligned}
\mathrm{tr}_{\,\mathfrak{g}}(I_a I_b I_c) &= -f^{\widetilde{y}}_{a\widetilde{x}} f^{\widetilde{z}}_{b\widetilde{y}} f^{\widetilde{x}}_{c\widetilde{z}} \\
&= -(\underbrace{f^{y}_{ax} f^{z}_{by} f^{x}_{cz} + f^{i}_{ax} f^{z}_{bi} f^{x}_{cz} + f^{y}_{ax} f^{i}_{by} f^{x}_{ci}}_{-\,\mathrm{tr}\,\mathfrak{m}} + \underbrace{f^{y}_{ai} f^{z}_{by} f^{i}_{cz}}_{-\,\mathrm{tr}\,\mathfrak{h}}) \\
&= -f^{y}_{ax} f^{z}_{by} f^{x}_{cz} + 3\,\mathrm{tr}_{\,\mathfrak{h}}(I_a I_b I_c).
\end{aligned} \tag{C.14}$$

Linearity of the trace yields

$$\begin{aligned}
-f^{y}_{[a|x} f^{z}_{|b]y} f^{x}_{cz} &= \frac{1}{2}\left(\mathrm{tr}_{\,\mathfrak{g}}([I_a, I_b] I_c) - 3\,\mathrm{tr}_{\,\mathfrak{h}}([I_a, I_b] I_c)\right) \\
&= \frac{1}{2} f^{d}_{ab}\left(\mathrm{tr}_{\,\mathfrak{g}}(I_d I_c) - 3\,\mathrm{tr}_{\,\mathfrak{h}}(I_d I_c)\right) \\
&= \frac{1}{2} f^{d}_{ab}\left(-\delta_{dc} + \frac{3}{2}(1-\alpha)\delta_{cd}\right) \\
&= \frac{1}{4}(1-3\alpha) f^{c}_{ab}.
\end{aligned} \tag{C.15}$$

After renaming and rearranging indices, this leads to

$$2f^e_{c[a}f^e_{b]d}f^f_{cd} = \frac{1}{2}(1-3\alpha)f^f_{ab}, \tag{C.16}$$

and we find

$$\begin{aligned} f^e_{[ab}f^e_{cd]}f^f_{cd} &= f^e_{[ab}f^e_{cd]}f^f_{cd} \\ &= \frac{2}{3}f^e_{c[a}f^e_{b]d}f^f_{cd} + \frac{1}{3}f_{eab}f^e_{cd}f^f_{cd} \\ &= \frac{1}{3}\left(\frac{1}{2}(1-3\alpha)+\alpha\right)f^f_{ab} \\ &= \frac{1}{6}(1-\alpha)f^f_{ab}. \end{aligned} \tag{C.17}$$

This proves equation (6.26).

Note that these identities only hold if the coset space metric is proportional to δ_{ab}. On Sasakian manifolds, for example, the metric is only blockwise proportional to δ_{ab}, and the sums of structure constants are slightly different.

C.2 Yang-Mills Equation in Components

Let (M, g) be a Riemannian manifold of dimension d, $\mathcal{A}$ the gauge connection with curvature $\mathcal{F}$ in a vector bundle E over M and $Q \in \Omega^4(M)$ a four-form. The torsionful Yang-Mills equation on M reads

$$d * \mathcal{F} + [\mathcal{A}, *\mathcal{F}] + *\mathcal{H} \wedge \mathcal{F} = 0, \tag{C.18}$$

with some three-form $\mathcal{H}$.

We compute the Yang-Mills equation in components, considering each summand separately. We use capital indices $A = (1, \ldots, d)$ to label directions on the manifold and restrict the consideration to metrics with Euclidean signature. With

$$*\mathcal{F} = \frac{1}{2(d-2)!}\mathcal{F}^{AB}\epsilon_{ABM_1\cdots M_{d-2}}e^{M_1\cdots M_{d-2}}, \tag{C.19}$$

the first summand takes the following form:

$$\begin{aligned} d * \mathcal{F} = \frac{1}{2(d-2)!}\Big(&\partial_C\left(\mathcal{F}^{AB}\epsilon_{ABM_1\ldots M_{d-2}}\right)e^{CM_1\cdots M_{d-2}} \\ &+ \mathcal{F}^{AB}\epsilon_{ABM_1\cdots M_{d-2}}d(e^{M_1\cdots M_{d-2}})\Big). \end{aligned} \tag{C.20}$$

Let Γ^C_{AB} be a connection in TM with torsion T^C_{AB}. Then Cartan's structure equation (3.25) takes the form

$$de^A = \left(\frac{1}{2}T^A_{CD} - \Gamma^A_{CD}\right)e^{CD}, \tag{C.21}$$

and we find

$$\begin{aligned} d(e^{M_1 \dots M_{d-2}}) &= \sum_{P=1}^{d-2}(-1)^{(P-1)} e^{M_1} \wedge \dots \wedge de^{M_P} \wedge \dots \wedge e^{M_{d-2}} \\ &= \sum_{P=1}^{d-2}(-1)^{(P-1)} \left(\frac{1}{2}T^{M_P}_{CD} - \Gamma^{M_P}_{CD}\right) e^{CDM_1 \cdots \widehat{M_P} \cdots M_{d-2}}. \end{aligned} \tag{C.22}$$

The hat indicates one-forms that are omitted in the wedge product, such as for example $e^{A_1} \wedge e^{\widehat{A}_2} \wedge e^{A_3} = e^{A_1} \wedge e^{A_3}$. Inserting equation (C.22) into equation (C.20), using $\epsilon_{ABM_1 \cdots M_{d-2}} = \sqrt{|g|}\varepsilon_{ABM_1 \cdots M_{d-2}}$ and the fact that the flat Kronecker symbol $\varepsilon_{ABM_1 \cdots M_{d-2}}$ is a constant yields

$$\begin{aligned} d * \mathcal{F} =& \frac{1}{2(d-2)!}\left(\partial_C\left(\sqrt{|g|}\mathcal{F}^{AB}\right)\varepsilon_{ABM_1 \cdots M_{d-2}} e^{CM_1 \dots M_{d-2}} \right. \\ & \left. + \sum_{P=1}^{d-2}(-1)^{(P-1)}\mathcal{F}^{AB}\epsilon_{ABM_1 \cdots M_{d-2}}\left(\frac{1}{2}T^{M_P}_{CD} - \Gamma^{M_P}_{CD}\right)e^{CDM_1 \cdots \widehat{M_P} \cdots M_{d-2}}\right). \end{aligned} \tag{C.23}$$

To avoid confusion, it is useful to apply the Hodge star operator once again. To do so, we first note that the components in the single summands of the above form read as follows, assuming for simplicity that the metric has nonvanishing entries only on the diagonal. This assumption covers all cases we are interested in – in particular cones over coset spaces with Killing metric (2.19) and Sasakian manifolds with metric (9.2) – and can easily be generalized for more complicated metrics.

$$\begin{aligned} &(d * \mathcal{F})^{CM_1 \cdots M_{d-2}} \\ &\quad = g^{CC} g^{M_1 M_1} \cdots g^{M_{d-2} M_{d-2}} (d * \mathcal{F})_{CM_1 \cdots M_{d-2}} \\ &\quad = \frac{(d-1)!}{2!(d-2)!} g^{CC} \partial_C \left(\sqrt{|g|}\mathcal{F}^{AB}\right) \varepsilon_{ABM_1 \cdots M_{d-2}} g^{M_1 M_1} \cdots g^{M_{d-2} M_{d-2}}, \end{aligned} \tag{C.24}$$

$$\begin{aligned} &(d * \mathcal{F})^{CDM_1 \cdots \widehat{M_p} \cdots M_{d-2}} \\ &\quad = \frac{(d-1)!}{2(d-2)!}(-1)^{(P-1)}\mathcal{F}_{AB}\epsilon^{ABM_1 \cdots M_{d-2}}\left(\frac{1}{2}T^{CD}{}_{M_P} - \Gamma^{CD}{}_{M_P}\right). \end{aligned} \tag{C.25}$$

When rearranging indices, the upper index of T^C_{AB} and Γ^C_{AB} is always lowered such that it appears *behind* the lower two ones. This is important in the following, as not all quantities can a priori be assumed to have totally antisymmetric indices. Equation (C.23) now takes the form

$$
\begin{aligned}
d\mathcal{F} = \frac{1}{2(d-2)!} \Big(& g^{CC}\partial_C \left(\sqrt{|g|}\mathcal{F}^{AB}\right) \varepsilon_{ABM_1\cdots M_{d-2}} g^{M_1M_1}\cdots g^{M_{d-2}M_{d-2}} \epsilon_{CM_1\cdots M_{d-2}Q} \\
&+ \sum_{P=1}^{d-2} (-1)^{(P-1)} \mathcal{F}_{AB}\epsilon^{ABM_1\cdots M_{d-2}} \\
&\left(\frac{1}{2}T^{CD}{}_{M_P} - \Gamma^{CD}{}_{M_P}\right)\epsilon_{CDM_1\cdots\widehat{M_P}\cdots M_{d-2}Q}\Big) e^Q.
\end{aligned} \tag{C.26}
$$

The Levi-Civita tensors in the first summand cannot be contracted in the standard way. We find, by use of $\varepsilon_{ABM_1\cdots M_{d-2}}\varepsilon^{ABM_1\cdots M_{d-2}} = d!$ and the identity (B.8):

$$
\begin{aligned}
&\varepsilon_{ABM_1\cdots M_{d-2}} g^{M_1M_1}\cdots g^{M_{d-2}M_{d-2}}\epsilon_{CM_1\cdots M_{d-2}Q} \\
&\qquad = g_{AA}g_{BB}g^{AA}g^{BB}g^{M_1M_1}\cdots g^{M_{d-2}M_{d-2}}\varepsilon_{ABM_1\cdots M_{d-2}}\epsilon_{CM_1\cdots M_{d-2}Q} \\
&\qquad = \frac{g_{AA}g_{BB}}{|g|}\varepsilon_{ABM_1\cdots M_{d-2}}\epsilon_{CM_1\cdots M_{d-2}Q} \\
&\qquad = \frac{g_{AA}g_{BB}}{\sqrt{|g|}}\epsilon^{ABM_1\cdots M_{d-2}}\epsilon_{CM_1\cdots M_{d-2}Q} \\
&\qquad = \frac{g_{AA}g_{BB}}{\sqrt{|g|}}(-1)^{d-2}\epsilon^{ABM_1\cdots M_{d-2}}\epsilon_{CQM_1\ldots M_{d-2}} \\
&\qquad = \frac{g_{AA}g_{BB}}{\sqrt{|g|}}(-1)^{d-2}2!(d-2)!\delta^{AB}_{CQ}.
\end{aligned} \tag{C.27}
$$

With this identity and

$$
\begin{aligned}
\epsilon^{ABM_1\cdots M_{d-2}}\epsilon_{CDM_1\cdots\widehat{M_P}\cdots M_{d-2}Q} &= (-1)^{P+d-4}\epsilon^{ABM_PM_1\cdots\widehat{M_P}\cdots M_{d-2}}\epsilon_{CDQM_1\cdots\widehat{M_P}\cdots M_{d-2}} \\
&= (-1)^{P+d-4}3!(d-3)!\delta^{ABM_P}_{CDQ},
\end{aligned} \tag{C.28}
$$

equation (C.26) turns into

$$
\begin{aligned}
d\mathcal{F} = \Bigg(& (-1)^{d-2}\frac{g_{AA}g_{BB}g^{CC}}{\sqrt{|g|}}\partial_C\left(\sqrt{|g|}\mathcal{F}^{AB}\right)\delta^{AB}_{CQ} \\
&+ \sum_{P=1}^{d-2}(-1)^{d-5}\frac{3!(d-3)!}{2!(d-2)!}\mathcal{F}_{AB}\left(\frac{1}{2}T^{KL}{}_{M_P} - \Gamma^{KL}{}_{M_P}\right)\delta^{ABM_P}_{CDQ}\Bigg) e^Q.
\end{aligned} \tag{C.29}
$$

After evaluation of the sum and the antisymmetrization of indices, we find

$$\begin{aligned}
d\mathcal{F} = (-1)^{d-2} \Bigg(& \frac{g_{QQ}}{\sqrt{|g|}} \partial_C \left(\sqrt{|g|} \mathcal{F}^{CQ} \right) \\
& - \Bigg(\frac{1}{2} \left(\mathcal{F}_{CD} T^{CD}{}_Q - \mathcal{F}_{CQ} T^{CD}{}_D + \mathcal{F}_{DQ} T^{CD}{}_C \right) \\
& \quad + \left(\mathcal{F}_{CD} \Gamma^{CD}{}_Q - \mathcal{F}_{CQ} \Gamma^{CD}{}_D + \mathcal{F}_{DQ} \Gamma^{CD}{}_D \right) \Bigg) \Bigg) e^Q. \qquad \text{(C.30)}
\end{aligned}$$

The second summand $[\mathcal{A}, *\mathcal{F}]$ of the Yang-Mills equation takes the form

$$\begin{aligned}
[\mathcal{A}, *\mathcal{F}] &= \left[\mathcal{A}_A e^A, \frac{1}{2(d-2)!} \mathcal{F}^{CD} \epsilon_{CDM_1 \cdots M_{d-2}} e^{M_1 \cdots M_{d-2}} \right] \\
&= \frac{1}{2(d-2)!} [\mathcal{A}_A, \mathcal{F}^{CD}] \epsilon_{CDM_1 \cdots M_{d-2}} e^{AM_1 \cdots M_{d-2}}. \qquad \text{(C.31)}
\end{aligned}$$

Applying the Hodge star again and using

$$\begin{aligned}
\epsilon^{CDM_1 \cdots M_{d-2}} \epsilon_{AM_1 \cdots M_{d-2} Q} &= (-1)^{d-2} \epsilon^{CDM_1 \cdots M_{d-2}} \epsilon_{AQM_1 \cdots M_{d-2}} \\
&= (-1)^{d-2} 2(d-2)! \delta^{CD}_{AQ}, \qquad \text{(C.32)}
\end{aligned}$$

yields

$$\begin{aligned}
*[\mathcal{A}, *\mathcal{F}] &= \frac{1}{2(d-2)!} [\mathcal{A}^A, \mathcal{F}_{CD}] \epsilon^{CDM_1 \cdots M_{d-2}} \epsilon_{AM_1 \cdots M_{d-2} Q} e^Q \\
&= (-1)^{d-2} [\mathcal{A}^A, \mathcal{F}_{AQ}] e^Q. \qquad \text{(C.33)}
\end{aligned}$$

Let us proceed by computing the torsion term. The dual of the three-form $\mathcal{H}$ is given by

$$*\mathcal{H} = \frac{1}{3!(d-3)!} \mathcal{H}^{ABC} \epsilon_{ABCM_1 \cdots M_{d-3}} e^{M_1 \cdots M_{d-3}}. \qquad \text{(C.34)}$$

Taking the wedge product with $\mathcal{F}$, we obtain

$$*\mathcal{H} \wedge \mathcal{F} = \frac{1}{2 \cdot 3!(d-3)!} \mathcal{H}^{ABC} \mathcal{F}_{KL} \epsilon_{ABCM_1 \cdots M_{d-3}} e^{KLM_1 \cdots M_{d-3}}, \qquad \text{(C.35)}$$

which leaves us with the following components:

$$(*\mathcal{H} \wedge \mathcal{F})_{KLM_1 \cdots M_{d-3}} = \frac{1}{2 \cdot 3!(d-3)!} \mathcal{H}^{ABC} \mathcal{F}_{KL} \epsilon_{ABCM_1 \cdots M_{d-3}}. \qquad \text{(C.36)}$$

Applying the Hodge operator again therefore yields

$$\begin{aligned}
(\mathcal{H} \wedge \mathcal{F}) &= \frac{1}{2 \cdot 3!(d-3)!} \mathcal{H}_{ABC} \mathcal{F}^{KL} \epsilon^{ABCM_1 \cdots M_{d-3}} \epsilon_{KLM_1 \cdots M_{d-3} Q} e^Q \\
&= \frac{1}{2} (-1)^{d-3} \mathcal{H}_{ABC} \mathcal{F}^{KL} \delta^{ABC}_{KLQ} e^Q \\
&= \frac{1}{2} (-1)^{d-3} \mathcal{H}_{KLQ} \mathcal{F}^{KL} e^Q. \qquad \text{(C.37)}
\end{aligned}$$

After suitably renaming the indices, the entire Yang-Mills equation with torsion takes the form

$$\frac{g_{BB}}{\sqrt{|g|}}\partial_C\left(\sqrt{|g|}\mathcal{F}^{CB}\right) - \mathcal{F}^{CD}\left(\frac{1}{2}T_{CDB} - \Gamma_{CDB}\right) + \mathcal{F}^{C}{}_{B}\left(\frac{1}{2}T_{CD}{}^{D} - \Gamma_{CD}{}^{D}\right) - \mathcal{F}^{C}{}_{B}\left(\frac{1}{2}T_{DC}{}^{D} - \Gamma_{DC}{}^{D}\right) + [\mathcal{A}^A, F_{AB}] - \frac{1}{2}\mathcal{H}_{CDB}\mathcal{F}^{CD} = 0. \tag{C.38}$$

The following special cases of this equation will be of particular interest for us: first, let $M = \mathcal{Z}(N)$ be the cylinder over a G-structure manifold N and assume that the components of the metric g_N are coordinate-independent. Then the Yang-Mills equation becomes

$$\partial_C\mathcal{F}^{CB} - \mathcal{F}^{CD}\left(\frac{1}{2}T_{CD}{}^{B} - \Gamma_{CD}{}^{B}\right) + \mathcal{F}^{CB}\left(\frac{1}{2}T_{CD}{}^{D} - \Gamma_{CD}{}^{D}\right) - \mathcal{F}^{CB}\left(\frac{1}{2}T_{DC}{}^{D} - \Gamma_{DC}{}^{D}\right) + [\mathcal{A}_A, F^{AB}] - \frac{1}{2}\mathcal{H}_{CD}{}^{B}\mathcal{F}^{CD} = 0. \tag{C.39}$$

This equation is used in Chapter 9 to derive second-order equations on the cylinder over a Sasakian manifold.

Second, let $M = \mathcal{C}(G/H)$ be the cone over a coset space G/H with metric

$$g_{\mathcal{C}} = \gamma^2 e^{2\gamma\tau}(d\tau^2 + \delta_{ab}e^a e^b), \tag{C.40}$$

where small indices label directions on G/H. In this case, we have to distinguish whether the free index B becomes zero or nonzero. The Yang-Mills equation then leads to

$$\mathcal{F}^{CD}\left(\frac{1}{2}T_{CD}{}^{0} - \Gamma_{CD}{}^{0}\right) - \mathcal{F}^{c0}\left(\frac{1}{2}T_{cD}{}^{D} - \Gamma_{cD}{}^{D}\right) + \mathcal{F}^{c0}\left(\frac{1}{2}T_{Dc}{}^{D} - \Gamma_{Dc}{}^{D}\right) - [\mathcal{A}_a, F^{a0}] + \frac{1}{2}\mathcal{H}_{CD}{}^{0}\mathcal{F}^{CD} = 0 \tag{C.41}$$

for $B = 0$ and

$$\gamma^{-4}e^{-4\gamma\tau}\partial_0\mathcal{F}_{0b} + \gamma(d-4)\mathcal{F}^{0b} - \mathcal{F}^{CD}\left(\frac{1}{2}T_{CD}{}^{b} - \Gamma_{CD}{}^{b}\right) + \mathcal{F}^{Cb}\left(\frac{1}{2}T_{CD}{}^{D} - \Gamma_{CD}{}^{D}\right) - \mathcal{F}^{Cb}\left(\frac{1}{2}T_{DC}{}^{D} - \Gamma_{DC}{}^{D}\right) + [\mathcal{A}_a, F^{ab}] - \frac{1}{2}\mathcal{H}_{CD}{}^{b}\mathcal{F}^{CD} = 0 \tag{C.42}$$

for $B \neq 0$. These equations are used in Chapter 8.

C.3 Details for the Yang-Mills Equation on Sasakian Manifolds

In this section, we collect some details about identities that are used in Chapter 9.

Sum of structure constants

Let M be a Sasakian manifold of dimension $(2m+1)$ with structure group $SU(m)$ and metric

$$g_M = e^1 e^1 + \frac{2m}{m+1}\delta_{ab}e^{ab}. \tag{C.43}$$

Let $\mathbb{R}\times M$ be the cylinder with structure group $SU(m+1)$. Then the Lie algebras corresponding to the structure groups admit a splitting $\mathfrak{su}(m+1) = \mathfrak{su}(m) \oplus \mathfrak{m}$, as described in Chapter 9. We use indices $\mu = (1, 2, \dots, \dim \mathfrak{m})$ to label the generators of $\mathfrak{m}$ and indices i, j for the remaining generators of $\mathfrak{su}(m)$. In this setup, the $SU(m+1)$-structure constants satisfy equation (9.21):

$$f^i_{ac} f^c_{ib} = \frac{2(m^2-1)}{m}\delta_{ab}. \tag{C.44}$$

To see this, note first that the components of the metric (C.43) take the form

$$(g_M)_{11} = 1, \qquad (g_M)_{ab} = \frac{2m}{m+1}\delta_{ab}. \tag{C.45}$$

The Killing form of $\mathfrak{su}(m+1)$ induces the following metric on $\mathfrak{m}$:

$$(g_K)_{\mu\nu} = f^{\widetilde{d}}_{\mu\widetilde{c}} f^{\widetilde{c}}_{\widetilde{d}\nu}. \tag{C.46}$$

The structure constants are normalized such that they satisfy

$$f^1_{ab} = 2P_{ab1}, \qquad f^b_{1a} = \frac{m+1}{m}P_{1ab}. \tag{C.47}$$

Hence, the Killing metric takes the following values:

$$(g_K)_{11} = f^{\widetilde{d}}_{1\widetilde{c}} f^{\widetilde{c}}_{\widetilde{d}1} = f^d_{1c} f^c_{d1} = \frac{2(m+1)^2}{m} =: X, \tag{C.48}$$

$$(g_K)_{ab} = 2(f^d_{a1} f^1_{db} + f^d_{ai} f^i_{db}) = 2\left(\frac{2(m+1)}{m}\delta_{ab} + f^d_{ai} f^i_{db}\right). \tag{C.49}$$

This metric matches the metric (C.43) up to rescaling of structure constants by the factor $\sqrt{X}$. We therefore find

$$(g_M)_{11} = \frac{1}{X}(g_K)_{11} = 1, \tag{C.50}$$

$$\begin{aligned}(g_M)_{ab} &= \frac{1}{X}(g_K)_{ab} \\ &= \frac{2}{X}(f^d_{a1}f^1_{db} + f^d_{ai}f^i_{db}) \\ &= \frac{2}{X}\left(\frac{2(m+1)}{m}\delta_{ab} + f^d_{ai}f^i_{db}\right) \\ &= \frac{2m}{m+1}\delta_{ab}.\end{aligned} \tag{C.51}$$

We conclude that both summands in $(g_M)_{ab}$ must be proportional to δ_{ab}, hence $f^d_{ai}f^i_{db} \overset{!}{=} \beta\delta_{ab}$ with some real parameter $\beta \in \mathbb{R}$. This leads to

$$\frac{2}{X}\left(\frac{2(m+1)}{m} + \beta\right)\delta_{ab} = \frac{2m}{m+1}\delta_{ab} \qquad \Rightarrow \qquad \beta = \frac{2(m^2-1)}{m} \tag{C.52}$$

and proves equation (C.44).

Action

The second-order equations (9.22) derived from the Yang-Mills equation on the cylinder over a Sasakian manifold are the equations of motion for the action

$$\begin{aligned} S &= \frac{m}{4(m+1)}\int_{\mathbb{R}\times M} \mathrm{tr}\left(\mathcal{F}\wedge *\mathcal{F} + 2\left(\frac{m}{m+1}\right)^2 \kappa d\tau \wedge *_M Q_M \wedge \mathcal{F}\wedge\mathcal{F}\right) \\ &= Vol(M) \times \int_{\mathbb{R}}\Bigg[-\frac{1}{2}(\dot{\chi}^2 + \dot{\psi}^2) - \left(\frac{m+1}{m}\right)^2 \\ &\qquad \left(\psi^2(1-\chi)^2 + m(1-m)(1-\kappa)\left(\frac{1}{2m}\psi^2 - 1\right)^2\right. \\ &\qquad \left. + m(1+\kappa(m-1))\left(\chi - \frac{1}{2m}\psi^2\right)^2\right)\Bigg] d\tau \end{aligned} \tag{C.53}$$

with potential

$$\begin{aligned} V(\chi,\psi) = \frac{1}{2}\left(\frac{m+1}{m}\right)^2 &\Big(((1+\kappa(m-1))m\chi^2 + (\kappa(1-m)-3)\,\chi\psi^2 \\ &+ \chi^2\psi^2 + (2-m+\kappa(m-1))\,\psi^2 + \frac{1}{4}\psi^4 + m(m-1)(1-\kappa)\Big). \end{aligned} \tag{C.54}$$

To see this, we compute the summands tr $(\mathcal{F} \wedge *\mathcal{F})$ and tr $(d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F})$ separately. For the first summand, we find

$$\begin{aligned}
\text{tr}\,(\mathcal{F} \wedge *\mathcal{F}) =& \frac{1}{2}\,\text{tr}\,(2\mathcal{F}_{0\mu}\mathcal{F}^{0\mu} + \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})Vol(\mathbb{R}\times M) \\
=& \frac{1}{2}\,\text{tr}\,\Bigg(2\mathcal{F}_{01}\mathcal{F}_{01} + \frac{m+1}{m}\mathcal{F}_{0a}\mathcal{F}_{0a} \\
&+ \frac{m+1}{m}\mathcal{F}_{1b}\mathcal{F}_{1b} + \left(\frac{m+1}{2m}\right)^2 \mathcal{F}_{ab}\mathcal{F}_{ab}\Bigg) Vol(\mathbb{R}\times M) \\
=& \frac{1}{2}\Bigg(-4\frac{m+1}{m}\left(\dot{\chi}^2 + \dot{\psi}^2\right) - 4\left(\frac{m+1}{m}\right)^3 \psi^2(1-\chi)^2 \\
&+ \left(\frac{m+1}{2m}\right)^2 \Bigg(\left(\frac{1}{2m}\psi^2 - 1\right)^2 f^i_{ab}f^j_{ab}f^n_{im}f^m_{jn} \\
&- 16(m+1)\left(\chi - \frac{1}{2m}\psi^2\right)^2\Bigg)\Bigg) Vol(\mathbb{R}\times M), \quad \text{(C.55)}
\end{aligned}$$

using $Vol(\mathbb{R}\times M) = \sqrt{|g_{\mathcal{Z}}|}d\tau \wedge e^{1\cdots(2m+1)}$, the components (9.20) of the curvature and the following explicit expressions for the trace of I_i, I_μ in the representation (9.4):

$$\text{tr}\,(I_1 I_1) = I^a_{10}I^0_{1a} + I^b_{1a}I^a_{1b} + I^0_{11}I^1_{10} + I^1_{10}I^0_{11} = -2\frac{m+1}{m}, \quad \text{(C.56)}$$

$$\text{tr}\,(I_i I_j) = I^b_{ia}I^a_{jb}, \quad \text{(C.57)}$$

$$\text{tr}\,(I_1 I_j) = 0, \quad \text{(C.58)}$$

$$\text{tr}\,(I_a I_a) = 2(I^b_{a0}I^0_{ab} + I^b_{a1}I^1_{ab}) = -8m \text{ (sum over } a). \quad \text{(C.59)}$$

The combination $f^i_{ab}f^j_{ab}f^n_{im}f^m_{jn}$ of structure constants in equation (C.55) can be simplified by use of the following relation. The commutator of two generators in the representation (9.4) takes the form

$$[I_a, I_b]^d_c = f^i_{ab}I^d_{ic} + f^1_{ab}I^d_{1c}. \quad \text{(C.60)}$$

Inserting the explicit expressions for I_1 and I_a leads to the identity

$$f^i_{ab}f^d_{ic} = \omega_{bc}\omega_{ad} - \omega_{ac}\omega_{bd} - \delta^c_a\delta^d_b + \delta^c_b\delta^d_a + \frac{2}{m}P_{ab1}\omega_{cd}. \quad \text{(C.61)}$$

We use this expression to rewrite the sum of structure constants in equation (C.55) and find

$$\begin{aligned}\sum_{a,b,c,d,i,j} f^i_{ab} f^j_{ab} f^n_{im} f^m_{jn} = \sum_{a,b,c,d,i,j} &\left(\omega_{bc}\omega_{ad} - \omega_{ac}\omega_{bd} - \delta^c_a\delta^d_b + \delta^c_b\delta^d_a + \frac{2}{m}\omega_{ab}\omega_{cd}\right) \\ &\left(\omega_{bd}\omega_{ac} - \omega_{ad}\omega_{bc} - \delta^d_a\delta^c_b + \delta^d_b\delta^c_a - \frac{2}{m}\omega_{ab}\omega_{cd}\right) \\ = \sum_{a,b,c,d,i,j} &\Big(2\omega_{bc}\omega_{ad}\omega_{bd}\omega_{ac} - 2\omega_{ac}\omega_{bd}\omega_{ac}\omega_{bd} + \frac{4}{m}\omega_{ab}\omega_{cd}\omega_{bd}\omega_{ac} \\ &- \frac{4}{m}\omega_{ab}\omega_{cd}\omega_{bc}\omega_{ad} - \frac{4}{m^2}\omega_{ab}\omega_{cd}\omega_{ab}\omega_{cd} \\ &+ \left(\frac{8}{m} - 4\right)\omega_{ab}\omega_{ab} + 2(\delta^d_a\delta^d_a - \delta^d_a\delta^c_b)\Big) \\ = 4m - 8m^2 + 8 + 8 - 16 + &\left(\frac{8}{m} - 4\right) 2m + 4m - 8m^2 \\ = 16(1 - m^2), & \qquad (C.62)\end{aligned}$$

using the fact that $\omega_{ab}\omega_{ab} = 2m$ and that only the components of ω_{ab} with $b = a+1$ or $b = a - 1$ are nonzero. To avoid confusion, the summation indices have been explicitly displayed at this point. Note that all indices are being summed over. Inserting this back into equation (C.55) yields

$$\begin{aligned}\mathrm{tr}\,(\mathcal{F} \wedge *\mathcal{F}) =& 4\frac{m+1}{m}\Bigg(-\frac{1}{2}\left(\dot\chi^2 + \dot\psi^2\right) - \left(\frac{m+1}{m}\right)^2 \psi^2(1-\chi)^2 \\ &+ (1-m^2)\frac{m+1}{m}\left(\frac{1}{2m}\psi^2 - 1\right)^2 \\ &- \frac{(m+1)^2}{m}\left(\chi - \frac{1}{2m}\psi^2\right)^2\Bigg) Vol(\mathbb{R} \times M) \\ =& 4\frac{m+1}{m}\Bigg(-\frac{1}{2}\left(\dot\chi^2 + \dot\psi^2\right) - \left(\frac{m+1}{m}\right)^2 \Big(\psi^2(1-\chi)^2 \\ &- (1-m)m\left(\frac{1}{2m}\psi^2 - 1\right)^2 \\ &+ m\left(\chi - \frac{1}{2m}\psi^2\right)^2\Big)\Bigg) Vol(\mathbb{R} \times M). \qquad (C.63)\end{aligned}$$

For the second summand in the action, note that $d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}$ is a form of top degree in on the cylinder $\mathcal{Z}(M)$. A convenient way to compute the components of this form is to apply the Hodge star operator. We find

$$\begin{aligned}
*_{\mathcal{Z}(M)}(d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}) &= *_M(*_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}) \\
&= \frac{1}{4n!(n-4)!} Q_{\mu\nu\rho\sigma} \mathcal{F}^{\alpha\beta} \mathcal{F}^{\gamma\delta} \epsilon^{\mu\nu\rho\sigma\xi_1 \cdots \xi_{n-4}} \epsilon_{\xi_1 \cdots \xi_{n-4}\alpha\beta\gamma\delta} \\
&= \frac{1}{4} Q_{\mu\nu\rho\sigma} \mathcal{F}^{\alpha\beta} \mathcal{F}^{\gamma\delta} \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} \\
&= \frac{1}{4} Q_{\mu\nu\rho\sigma} \mathcal{F}^{[\mu\nu} \mathcal{F}^{\rho\sigma]} \\
&= 3\omega_{\mu\nu}\omega_{\rho\sigma} \mathcal{F}^{[\mu\nu} \mathcal{F}^{\rho\sigma]},
\end{aligned} \tag{C.64}$$

using $n := 2m+1 = \dim M$ as well as $Q = \frac{1}{2}\omega \wedge \omega \Leftrightarrow Q_{\mu\nu\rho\sigma} = 4!\omega_{\mu\nu}\omega_{\rho\sigma}$. This result implies

$$d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F} = 3\omega_{ab}\omega_{cd} \mathcal{F}^{[ab} \mathcal{F}^{c]d} Vol(\mathbb{R} \times M). \tag{C.65}$$

We find

$$\begin{aligned}
&\mathrm{tr}\,(d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}) \\
&\quad = 3\omega_{ab}\omega_{cd}\,\mathrm{tr}\,(\mathcal{F}^{[ab}\mathcal{F}^{c]d}) Vol(\mathbb{R} \times M) \\
&\quad = \left(\frac{m+1}{2m}\right)^4 \left(\left(\frac{1}{2m}\psi^2 - 1\right)^2 3\omega_{ab}\omega_{cd} f^i_{[ab} f^j_{c]d} f^n_{im} f^m_{jn} \right. \\
&\qquad\qquad \left. - 8\frac{m+1}{m}\left(\chi - \frac{1}{2m}\psi^2\right)^2 3\omega_{ab}\omega_{cd} P_{[ab|1|} P_{c]d1} \right) Vol(\mathbb{R} \times M) \\
&\quad = \left(\frac{m+1}{2m}\right)^4 \left(2\left(\frac{1}{2m}\psi^2 - 1\right)^2 \omega_{ab}\omega_{cd} f^i_{bc} f^j_{ad} f^n_{im} f^m_{jn} \right. \\
&\qquad\qquad \left. - 32(m^2-1)\left(\chi - \frac{1}{2m}\psi^2\right)^2 \right) Vol(\mathbb{R} \times M),
\end{aligned} \tag{C.66}$$

by use of $f^1_{ab} f^i_{ab} = 0$ and $3\omega_{ab}\omega_{cd} P_{[ab|1|} P_{c]d1} = 4m(m-1)$. The sum of structure constants simplifies to

$$\omega_{ab}\omega_{cd} f^i_{[ab} f^j_{c]d} f^n_{im} f^m_{jn} = 16(m^2-1). \tag{C.67}$$

This identity is proven by writing the structure constants in terms of equation (C.61) and evaluating all sums explicitly. As the computation follows the same pattern as the derivation of equation (C.62), we do not present the details here. We find

$$\begin{aligned}
&\mathrm{tr}\,(d\tau \wedge *_M Q_M \wedge \mathcal{F} \wedge \mathcal{F}) \\
&\quad = \left(\frac{m+1}{2m}\right)^4 32(m^2-1) \left(\left(\frac{1}{2m}\psi^2 - 1\right)^2 - \left(\chi - \frac{1}{2m}\psi^2\right)^2 \right) Vol(\mathbb{R} \times M).
\end{aligned} \tag{C.68}$$

Now the identities (C.63) and (C.68) can be inserted into the action. This leads to the result (C.53), taking into account that the volume form on the cylinder satisfies $Vol(\mathbb{R} \times M) = d\tau \wedge Vol(M)$.

Eigenvalues of the Hesse Matrix

Let us once again consider the potential (9.24):

$$V(\chi, \psi) = \frac{1}{2}\left(\frac{m+1}{m}\right)^2 \Big(((1+\kappa(m-1))m\chi^2 + (\kappa(1-m)-3)\,\chi\psi^2 \\ +\chi^2\psi^2 + (2-m+\kappa(m-1))\,\psi^2 + \frac{1}{4}\psi^4 + m(m-1)(1-\kappa)\Big)\,. \quad \text{(C.69)}$$

The critical points (χ, ψ) of V that satisfy $\partial_\chi V = \partial_\psi V = 0$ are listed in equation (9.26), and the eigenvalues of the matrix

$$\begin{pmatrix} \frac{\partial^2 V}{\partial \chi^2} & \frac{\partial^2 V}{\partial \chi \partial \psi} \\ \frac{\partial^2 V}{\partial \psi \partial \chi} & \frac{\partial^2 V}{\partial \psi^2} \end{pmatrix} \quad \text{(C.70)}$$

have been presented in Table 6. The eigenvalues at the critical point $(\chi_2, \psi_2) = (1, \pm\sqrt{2m})$ need a more detailed discussion. They are given by

$$(\lambda_1, \lambda_2) = \Bigg(\frac{1}{2}\left(\frac{m+1}{m}\right)^2 \Big((5+\kappa(m+1))m + (1+\kappa(m-1))\sqrt{m(8+m)}\Big)\,, \\ \frac{1}{2}\left(\frac{m+1}{m}\right)^2 \Big((5+\kappa(m-1))m - (1+\kappa(m-1))\sqrt{m(8+m)}\Big)\Bigg)\,. \quad \text{(C.71)}$$

λ_1 is greater than zero for

$$\kappa > \kappa_+ := -\frac{5m + \sqrt{m(8+m)}}{m(m+1) + (m-1)\sqrt{m(8+m)}} \quad \text{(C.72)}$$

and smaller than zero otherwise. λ_2 is greater than zero for

$$\kappa < \kappa_- := \frac{-5m + \sqrt{m(8+m)}}{m(m-1) - (m-1)\sqrt{m(8+m)}} \quad \text{(C.73)}$$

and smaller otherwise. We have $\kappa_- > \kappa_+$ for any positive integer value of $m > 1$. The extremum of the potential at $(1, \pm\sqrt{2m})$ is therefore

$$\begin{array}{llll} 1) & \text{a saddle} & \text{for} & \kappa > \kappa_-, \\ 2) & \text{indefinite} & \text{for} & \kappa = \kappa_-, \\ 3) & \text{a minimum} & \text{for} & \kappa_- > \kappa > \kappa_+, \\ 4) & \text{indefinite} & \text{for} & \kappa = \kappa_+, \\ 5) & \text{a saddle} & \text{for} & \kappa_+ > \kappa. \end{array} \tag{C.74}$$

This observation is in agreement with the remaining cases listed in Table 6: since $\kappa_+ > \frac{3}{1-m}$, we find one positive and one negative eigenvalue for (χ_4, ψ_4).

We can expect Yang-Mills solutions when the extrema at $(1, \pm\sqrt{2m})$ are minima, i. e. for $\kappa_- > \kappa > \kappa_+$ (in particular for $\kappa = 1$), or saddle points, and dyon solutions when they are saddle points. As λ_1 and λ_2 do not simultaneously become smaller than zero for any fixed value of κ, the critical points never become maxima.

Bibliography

References in Order of Appearance

[1] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, "The Heterotic String," Phys. Rev. Lett. **54** (1985) 502.

[2] M. Green, J. H. Schwarz and E. Witten, "Superstring Theory," Volume 1, Cambridge University Press (1987).

[3] D. Lüst and R. Blumenhagen, "Basic Concepts of String Theory," Springer Verlag Berlin (1987).

[4] T. Kaluza, "On the Problem of Unity in Physics," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921** (1921) 966.

[5] O. Klein, "Quantum Theory and Five-Dimensional Theory of Relativity," Z. Phys. **37** (1926) 895 [Surveys High Energ. Phys. **5** (1986) 241].

[6] M. Grana, "Flux Compactifications in String Theory: A Comprehensive Review," Phys. Rept. **423** (2006) 910, arXiv:hep-th/0509003.

[7] R. Blumenhagen, B. Kors, D. Lüst and S. Stieberger, "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes," Phys. Rept. **445** (2007) 1, arXiv:hep-th/0610327.

[8] M. R. Douglas and S. Kachru, "Flux Compactification," Rev. Mod. Phys. **79** (2007) 733, arXiv:hep-th/0610102.

[9] A. Strominger, "Superstrings with Torsion," Nucl. Phys. B **274** (1986) 253.

[10] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, "Pseudoparticle Solutions of the Yang-Mills Equations," Phys. Lett. B **59** (1975) 85 .

[11] R. Rajaraman, "Solitons And Instantons. An Introduction To Solitons And Instantons In Quantum Field Theory," Amsterdam, Netherlands: North-Holland (1982).

[12] N. S. Manton and P. Sutcliffe, "Topological Solitons," Cambridge University Press (2004).

[13] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, "First Order Equations for Gauge Fields in Spaces of Dimension Greater Than Four," Nucl. Phys. B **214** (1983) 452.

[14] T. J. Willmore and N. J. Hitchin, "Global Riemannian Geometry," Ellis Horwood Ltd. (1984).

[15] K. P. Gemmer, A. S. Haupt, O. Lechtenfeld, C. Nölle and A. D. Popov, "Heterotic String Plus Five-Brane Systems with Asymptotic AdS_3," Adv. Theor. Math. Phys. **17** (2013) 771, arXiv:1202.5046.

[16] R. S. Ward, "Completely Solvable Gauge Field Equations in Dimension Greater Than Four," Nucl. Phys. B **236** (1984) 381.

[17] D. B. Fairlie and J. Nuyts, "Spherically Symmetric Solutions of Gauge Theories in Eight Dimensions," J. Phys. A **17** (1984) 2867.

[18] S. Fubini and H. Nicolai, "The Octonionic Instanton," Phys. Lett. B **155** (1985) 369.

[19] A. D. Popov, "Antiselfdual Solutions of the Yang-Mills Equations in $4n$-Dimensions," Mod. Phys. Lett. A **7** (1992) 2077.

[20] E. Corrigan, P. Goddard and A. Kent, "Some Comments on the ADHM Construction in $4k$-Dimensions," Commun. Math. Phys. **100** (1985) 1.

[21] J. Broedel, T. A. Ivanova and O. Lechtenfeld, "Construction of Noncommutative Instantons in $4k$ Dimensions," Mod. Phys. Lett. A **23** (2008) 179, arXiv:hep-th/0703009.

[22] J. A. Harvey and A. Strominger, "Octonionic Superstring Solitons," Phys. Rev. Lett. **66** (1991) 549.

[23] T. A. Ivanova, O. Lechtenfeld, A. D. Popov and T. Rahn, "Instantons and Yang-Mills Flows on Coset Spaces," Lett. Math. Phys. **89** (2009) 231, arXiv:0904.0654.

[24] D. Harland, T. A. Ivanova, O. Lechtenfeld and A. D. Popov, "Yang-Mills Flows on Nearly-Kähler Manifolds and G_2-Instantons," Commun. Math. Phys. **300** (2010) 185, arXiv:0909.2730.

[25] D. Harland and A. D. Popov, "Yang-Mills Fields in Flux Compactifications on Homogeneous Manifolds with $SU(4)$-Structure," JHEP **1202** (2012) 107, arXiv:1005.2837.

[26] I. Bauer, T. A. Ivanova, O. Lechtenfeld and F. Lubbe, "Yang-Mills Instantons and Dyons on Homogeneous G_2-Manifolds," JHEP **1010** (2010) 044, arXiv:1006.2388.

[27] A. S. Haupt, T. A. Ivanova, O. Lechtenfeld and A. D. Popov, "Chern-Simons Flows on Aloff-Wallach Spaces and $Spin(7)$-Instantons," Phys. Rev. D **83** (2011) 105028, arXiv:1104.5231.

[28] K. P. Gemmer, O. Lechtenfeld, C. Nölle and A. D. Popov, "Yang-Mills Instantons on Cones and Sine-Cones over Nearly-Kähler Manifolds," JHEP **1109** (2011) 103, arXiv:1108.3951.

[29] D. Harland and C. Nölle, "Instantons and Killing Spinors," JHEP **1203** (2012) 082, arXiv:1109.3552.

[30] M. Berger, "Sur les Groupes d'Holonomie Homogènes de Variétés à Connexion Affine et des Variétés Riemanniennes," Bulletin de la Société Mathématique de France 83 (1955): 279-330, http://eudml.org/doc/86895.

[31] C. Bär, "Real Killing Spinors and Holonomy," Communications in Mathematical Physics, June (II) 1993, Volume 154, Issue 3, pp 509-521.

[32] S. Chiossi and S. Salamon, "The Intrinsic Torsion of $SU(3)$ and $G(2)$ Structures," arXiv:math/0202282.

[33] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, "A New Infinite Class of Sasaki-Einstein Manifolds," Adv. Theor. Math. Phys. **8** (2006) 987, arXiv:hep-th/0403038.

[34] D. Cassani, G. Dall'Agata and A. F. Faedo, "Type IIB Supergravity on Squashed Sasaki-Einstein Manifolds," JHEP **1005** (2010) 094, arXiv:1003.4283.

[35] M. Klaput, A. Lukas and C. Matti, "Bundles over Nearly-Kähler Homogeneous Spaces in Heterotic String Theory," JHEP **1109** (2011) 100, arXiv:arXiv:1107.3573.

[36] K. Becker, M. Becker and J. H. Schwarz, "String Theory and M-Theory," Cambridge University Press (2007).

[37] M. Green, J. H. Schwarz and E. Witten, "Superstring Theory," Volume 2, Cambridge University Press (1987).

[38] K. Becker and S. Sethi, "Torsional Heterotic Geometries," Nucl. Phys. B **820** (2009) 1, arXiv:0903.3769.

[39] C. P. Boyer and K. Galicki, "Sasakian Geometry, Holonomy, and Supersymmetry," arXiv:math/0703231.

[40] A. Arvanitoyeorgos, "An Introduction to Lie Groups and the Geometry of Homogeneous Spaces," American Mathematical Society (2003).

[41] H. Baum, "Eichfeldtheorie," Springer Verlag Berlin (2009).

[42] M. Nakahara, "Geometry, Topology and Physics," Boca Raton, USA: Taylor & Francis (2003).

[43] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry," Volume 1, John Wiley (1963).

[44] D. Joyce, "Compact Manifolds with Special Holonomy," Oxford University Press, Oxford, 2000.

[45] F. Müller-Hoissen and R. Stuckl, "Coset Spaces and Ten-Dimensional Unified Theories," Class. Quant. Grav. **5** (1988) 27.

[46] C. Nölle, "Homogeneous Heterotic Supergravity Solutions with Linear Dilaton," J. Phys. A **45** (2012) 045402, arXiv:1011.2873.

[47] C. P. Boyer and K. Galicki, “Sasakian Geometry,” Oxford University Press (2008).

[48] S. Salamon, “Riemannian Geometry and Holonomy Groups,” Vol. 201 of Pitman Research Notes in Mathematics, Longman, Harlow (1989).

[49] F. Schulte-Hengesbach, “Half-Flat Structures on Lie Groups,” PhD thesis, Universität Hamburg (2010).

[50] R. Cleyton and A. Swann, “Torsion and the Einstein Equations,” Proceedings of the Workshop on Special Geometric Structures in String Theory, Bonn (2001).

[51] D. V. Alexeevskii, “Riemannian Spaces with Unusual Holonomy Groups,” Funkcional. Anal. i Prilozen 2 (1968), no. 2, 1-10. MR 37 6868.

[52] J. Sparks, “Sasaki-Einstein Manifolds,” Surveys Diff. Geom. **16** (2011) 265, arXiv:arXiv:1004.2461.

[53] A. Fino, L. Vezzoni and A. Andrada, “A Class of Sasakian 5-Manifolds,” arXiv:0807.1800.

[54] D. Conti and A. Fino, “Calabi-Yau Cones from Contact Reduction,” arXiv:0710.4441.

[55] T. A. Ivanova and A. D. Popov, “Instantons on Special Holonomy Manifolds,” Phys. Rev. D **85** (2012) 105012, arXiv:1203.2657.

[56] D. Tong, “TASI Lectures on Solitons: Instantons, Monopoles, Vortices and Kinks,” arXiv:hep-th/0509216.

[57] C. Nölle, “Heterotic Supergravity on Manifolds with Killing Spinors,” PhD thesis, Leibniz Universität Hannover (2012).

[58] S. K. Donaldson, “Anti Self-Dual Yang-Mills Connections Over Complex Algebraic Surfaces And Stable Vector Bundles,” Proc. Lond. Math. Soc. **50** (1985) 1.

[59] K. Uhlenbeck and S. T. Yau, “On the Existence of Hermitian-Yang-Mills Connections in Stable Vector Bundles,” Comm. Pure Appl. Math., 39: S257-S293.

[60] M. Larfors, “Revisiting Toric $SU(3)$-Structures,” Fortsch. Phys. **61** (2013) 1031 Fortsch. Phys. 61 (2013) 1031-1055, arXiv:1309.2953.

[61] M. Klaput, A. Lukas, C. Matti and E. E. Svanes, "Moduli Stabilising in Heterotic Nearly-Kähler Compactifications," JHEP **1301** (2013) 015, arXiv:1210.5933.

[62] K. Becker, D. Robbins and E. Witten, "The α' Expansion On A Compact Manifold Of Exceptional Holonomy," JHEP **1406** (2014) 051, arXiv:1404.2460.

[63] A. Chatzistavrakidis and G. Zoupanos, "Dimensional Reduction of the Heterotic String over Nearly-Kähler manifolds," JHEP **0909** (2009) 077, arXiv:0905.2398.

[64] S. Donaldson and E. Segal, "Gauge Theory in Higher Dimensions, II," arXiv:0902.3239.

[65] L. Rey-Bellet, "Differential Equations and Dynamical Systems," Classnotes for Math 645, University of Massachusetts, http://people.math.umass.edu/l̃r7q/ps_files/teaching/math645/ode.pdf.

[66] B.-V. Matioc, "Qualitative Theorie Gewöhnlicher Differentialgleichungen," Lecture Notes, Leibniz Universität Hannover (2014).

[67] A. S. Haupt, "M-theory Calabi-Yau Quantum Mechanics," arXiv:0911.5710.

[68] P. Koerber, "Lectures on Generalized Complex Geometry for Physicists," Fortsch. Phys. **59** (2011) 169, arXiv:1006.1536.

[69] S. Karigiannis, "What is... a G_2 Manifold?," Notices of the American Mathematical Society 58, 580-581 (2011), arXiv:1104.2296

[70] S. Karigiannis, "Flows of G_2 Structures, I," Quarterly Journal of Mathematics 60, 487-522 (2009), arXiv:math/0702077

[71] M. Fernández and A. Gray, "Riemannian Manifolds with Structure Group G_2," Ann. Mat. Pura Appl. 132 (1982), 19-45.

[72] I. Agricola, "The Srni Lectures on Non-Integrable Geometries with Torsion," arXiv:math/0606705

[73] O. Lechtenfeld, A. D. Popov and R. J. Szabo, "SU(3)-Equivariant Quiver Gauge Theories and Nonabelian Vortices," JHEP **0808** (2008) 093, arXiv:0806.2791.

[74] S. Bunk, O. Lechtenfeld, A. D. Popov and M. Sperling, "Instantons on Conical Half-Flat 6-Manifolds," JHEP **1501** (2015) 030,arXiv:1409.0030.

[75] T. Rahn, "Yang-Mills Configurations on Coset Spaces," Master's thesis, Leibniz Universität Hannover (2009).

[76] F. Lubbe, "Yang-Mills Solutions in Heterotic Flux Compactifications," Master's thesis, Leibniz Universität Hannover (2010).

[77] A. Elias-Zuniga, "Analytical Solution of the Damped Helmholtz-Duffing Equation," Appl. Math. Lett., 25 (12)(2012), pp. 2349-2353.

[78] I. Kovacic and M. Brennan, "The Duffing Equation," John Wiley (2011).

[79] W. Han and M. Petyt, "Geometrically Nonlinear Vibration Analysis of Thin, Rectangular Plates Using the Hierarchical Finite Element Method: 1st Mode of Laminated Plates and Higher Modes of Isotropic and Laminated Plates," Comput Struct. 63(2) (1997) 295-308.

[80] S. Lai, J. Harrington, Y. Xiang and K. Chow, "Accurate Analytical Perturbation Approach for Large Amplitude Vibration of Functionally Graded Beams," Int. J. Non-Linear Mech. (2011).

[81] Y. Chen and X. Lin, "Several Numerical Solution Techniques for Nonlinear Eardrum-Type Oscillations," J Sound Vib., 296 (2006), pp. 1059-1067.

[82] G. Duffing, "Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung," Vieweg (1918).

[83] S. Parthasarathy and M. Lakshmanan, "On the Exact Solution of the Duffing Oscillator," J. Sound Vib. **137**, 523.

[84] M. Tormählen, "Yang-Mills Solutions and Dyons on Cylinders over Coset Spaces with Sasakian Structure," arXiv:1412.7069.

[85] W. Schwalm, "Elliptic Functions sn, cn, dn, as Trigonometry," http://www.und.edu/instruct/schwalm/MAA_Presentation_10-02/handout.pdf.

[86] N. S. Manton and T. M. Samols, "Sphalerons On A Circle," Phys. Lett. B **207** (1988) 179.

[87] T. A. Ivanova and O. Lechtenfeld, "Yang-Mills Instantons and Dyons on Group Manifolds," Phys. Lett. B **670** (2008) 91, arXiv:0806.0394.

[88] S. Vandoren, "Lectures on Riemannian Geometry, Part II: Complex Manifolds."

[89] W. Merry, "Complex Manifolds," http://www.merry.io/resources/complexIII.pdf.

[90] A. Newlander and L. Nirenberg, "Complex Analytic Coordinates in Almost Complex Manifolds," Ann. of Math. (2) 65 (1957), 391404. MR 19,577a.

References in Alphabetical Order

[72] I. Agricola, "The Srni Lectures on Non-Integrable Geometries with Torsion," arXiv:math/0606705

[51] D. V. Alexeevskii, "Riemannian Spaces with Unusual Holonomy Groups," Funkcional. Anal. i Prilozen 2 (1968), no. 2, 1-10. MR 37 6868.

[53] A. Andrada, A. Fino and L. Vezzoni, "A Class of Sasakian 5-Manifolds," arXiv:0807.1800.

[40] A. Arvanitoyeorgos, "An Introduction to Lie Groups and the Geometry of Homogeneous Spaces," American Mathematical Society (2003).

[31] C. Bär, "Real Killing Spinors and Holonomy," Communications in Mathematical Physics, June (II) 1993, Volume 154, Issue 3, pp 509-521.

[26] I. Bauer, T. A. Ivanova, O. Lechtenfeld and F. Lubbe, "Yang-Mills Instantons and Dyons on Homogeneous G_2-Manifolds," JHEP **1010** (2010) 044, arXiv:1006.2388.

[41] H. Baum, "Eichfeldtheorie," Springer Verlag Berlin (2009).

[10] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, "Pseudoparticle Solutions of the Yang-Mills Equations," Phys. Lett. B **59** (1975) 85 .

[36] K. Becker, M. Becker and J. H. Schwarz, "String Theory and M-Theory," Cambridge University Press (2007).

[62] K. Becker, D. Robbins and E. Witten, "The α' Expansion On A Compact Manifold Of Exceptional Holonomy," JHEP **1406** (2014) 051, arXiv:1404.2460.

[38] K. Becker and S. Sethi, "Torsional Heterotic Geometries," Nucl. Phys. B **820** (2009) 1, arXiv:0903.3769.

[30] M. Berger, "Sur les Groupes d'Holonomie Homogènes de Variétés à Connexion Affine et des Variétés Riemanniennes," Bulletin de la Société Mathématique de France 83 (1955): 279-330, http://eudml.org/doc/86895.

[7] R. Blumenhagen, B. Kors, D. Lüst and S. Stieberger, "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes," Phys. Rept. **445** (2007) 1, arXiv:hep-th/0610327.

[3] R. Blumenhagen and D. Lüst, "Basic Concepts of String Theory," Springer Verlag Berlin (1987).

[47] C. P. Boyer and K. Galicki, "Sasakian Geometry," Oxford University Press (2008).

[39] C. P. Boyer and K. Galicki, "Sasakian Geometry, Holonomy, and Supersymmetry," arXiv:math/0703231.

[21] J. Broedel, T. A. Ivanova and O. Lechtenfeld, "Construction of Noncommutative Instantons in $4k$ Dimensions," Mod. Phys. Lett. A **23** (2008) 179, arXiv:hep-th/0703009.

[74] S. Bunk, O. Lechtenfeld, A. D. Popov and M. Sperling, "Instantons on Conical Half-Flat 6-Manifolds," JHEP **1501** (2015) 030,arXiv:1409.0030.

[34] D. Cassani, G. Dall'Agata and A. F. Faedo, "Type IIB Supergravity on Squashed Sasaki-Einstein Manifolds," JHEP **1005** (2010) 094, arXiv:1003.4283.

[63] A. Chatzistavrakidis and G. Zoupanos, "Dimensional Reduction of the Heterotic String over Nearly-Kähler manifolds," JHEP **0909** (2009) 077, arXiv:0905.2398.

[81] Y. Chen and X. Lin, "Several Numerical Solution Techniques for Nonlinear Eardrum-Type Oscillations," J Sound Vib., 296 (2006), pp. 1059-1067.

[32] S. Chiossi and S. Salamon, "The Intrinsic Torsion of $SU(3)$ and $G(2)$ Structures," arXiv:math/0202282.

[50] R. Cleyton and A. Swann, "Torsion and the Einstein Equations," Proceedings of the Workshop on Special Geometric Structures in String Theory, Bonn (2001).

[54] D. Conti and A. Fino, "Calabi-Yau Cones from Contact Reduction," arXiv:0710.4441.

[13] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, "First Order Equations for Gauge Fields in Spaces of Dimension Greater Than Four," Nucl. Phys. B **214** (1983) 452.

[20] E. Corrigan, P. Goddard and A. Kent, "Some Comments on the ADHM Construction in $4k$-Dimensions," Commun. Math. Phys. **100** (1985) 1.

[58] S. K. Donaldson, "Anti Self-Dual Yang-Mills Connections Over Complex Algebraic Surfaces And Stable Vector Bundles," Proc. Lond. Math. Soc. **50** (1985) 1.

[64] S. Donaldson and E. Segal, "Gauge Theory in Higher Dimensions, II," arXiv:0902.3239.

[8] M. R. Douglas and S. Kachru, "Flux Compactification," Rev. Mod. Phys. **79** (2007) 733, arXiv:hep-th/0610102.

[82] G. Duffing, "Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung," Vieweg (1918).

[77] A. Elias-Zuniga, "Analytical Solution of the Damped Helmholtz-Duffing Equation," Appl. Math. Lett., 25 (12)(2012), pp. 2349-2353.

[17] D. B. Fairlie and J. Nuyts, "Spherically Symmetric Solutions of Gauge Theories in Eight Dimensions," J. Phys. A **17** (1984) 2867.

[71] M. Fernández and A. Gray, "Riemannian Manifolds with Structure Group G_2," Ann. Mat. Pura Appl. 132 (1982), 19-45.

[18] S. Fubini and H. Nicolai, "The Octonionic Instanton," Phys. Lett. B **155** (1985) 369.

[33] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, "A New Infinite Class of Sasaki-Einstein Manifolds," Adv. Theor. Math. Phys. **8** (2006) 987, arXiv:hep-th/0403038.

[15] K. P. Gemmer, A. S. Haupt, O. Lechtenfeld, C. Nölle and A. D. Popov, "Heterotic String Plus Five-Brane Systems with Asymptotic AdS_3," Adv. Theor. Math. Phys. **17** (2013) 771, arXiv:1202.5046.

[28] K. P. Gemmer, O. Lechtenfeld, C. Nölle and A. D. Popov, "Yang-Mills Instantons on Cones and Sine-Cones over Nearly-Kähler Manifolds," JHEP **1109** (2011) 103, arXiv:1108.3951.

[6] M. Grana, "Flux Compactifications in String Theory: A Comprehensive Review," Phys. Rept. **423** (2006) 910, arXiv:hep-th/0509003.

[2] M. Green, J. H. Schwarz and E. Witten, "Superstring Theory," Volume 1, Cambridge University Press (1987).

[37] M. Green, J. H. Schwarz and E. Witten, "Superstring Theory," Volume 2, Cambridge University Press (1987).

[1] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, "The Heterotic String," Phys. Rev. Lett. **54** (1985) 502.

[79] W. Han and M. Petyt, "Geometrically Nonlinear Vibration Analysis of Thin, Rectangular Plates Using the Hierarchical Finite Element Method: 1st Mode of Laminated Plates and Higher Modes of Isotropic and Laminated Plates," Comput Struct. 63(2) (1997) 295-308.

[22] J. A. Harvey and A. Strominger, "Octonionic Superstring Solitons," Phys. Rev. Lett. **66** (1991) 549.

[24] D. Harland, T. A. Ivanova, O. Lechtenfeld and A. D. Popov, "Yang-Mills Flows on Nearly-Kähler Manifolds and G_2-Instantons," Commun. Math. Phys. **300** (2010) 185, arXiv:0909.2730.

[29] D. Harland and C. Nölle, "Instantons and Killing Spinors," JHEP **1203** (2012) 082, arXiv:1109.3552.

[25] D. Harland and A. D. Popov, "Yang-Mills Fields in Flux Compactifications on Homogeneous Manifolds with $SU(4)$-Structure," JHEP **1202** (2012) 107, arXiv:1005.2837.

[67] A. S. Haupt, "M-theory Calabi-Yau Quantum Mechanics," arXiv:0911.5710.

[27] A. S. Haupt, T. A. Ivanova, O. Lechtenfeld and A. D. Popov, "Chern-Simons Flows on Aloff-Wallach Spaces and $Spin(7)$-Instantons," Phys. Rev. D **83** (2011) 105028, arXiv:1104.5231.

[87] T. A. Ivanova and O. Lechtenfeld, "Yang-Mills Instantons and Dyons on Group Manifolds," Phys. Lett. B **670** (2008) 91, arXiv:0806.0394.

[23] T. A. Ivanova, O. Lechtenfeld, A. D. Popov and T. Rahn, "Instantons and Yang-Mills Flows on Coset Spaces," Lett. Math. Phys. **89** (2009) 231, arXiv:0904.0654.

[55] T. A. Ivanova and A. D. Popov, "Instantons on Special Holonomy Manifolds," Phys. Rev. D **85** (2012) 105012, arXiv:1203.2657.

[44] D. Joyce, "Compact Manifolds with Special Holonomy," Oxford University Press, Oxford, 2000.

[4] T. Kaluza, "On the Problem of Unity in Physics," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921** (1921) 966.

[70] S. Karigiannis, "Flows of G_2 Structures, I," Quarterly Journal of Mathematics 60, 487-522 (2009), arXiv:math/0702077

[69] S. Karigiannis, "What is... a G_2 Manifold?," Notices of the American Mathematical Society 58, 580-581 (2011), arXiv:1104.2296

[35] M. Klaput, A. Lukas and C. Matti, "Bundles over Nearly-Kähler Homogeneous Spaces in Heterotic String Theory," JHEP **1109** (2011) 100, arXiv:arXiv:1107.3573.

[61] M. Klaput, A. Lukas, C. Matti and E. E. Svanes, "Moduli Stabilising in Heterotic Nearly-Kähler Compactifications," JHEP **1301** (2013) 015, arXiv:1210.5933.

[5] O. Klein, "Quantum Theory and Five-Dimensional Theory of Relativity," Z. Phys. **37** (1926) 895 [Surveys High Energ. Phys. **5** (1986) 241].

[43] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry," Volume 1, John Wiley (1963).

[78] I. Kovacic and M. Brennan, "The Duffing Equation," John Wiley (2011).

[68] P. Koerber, "Lectures on Generalized Complex Geometry for Physicists," Fortsch. Phys. **59** (2011) 169, arXiv:1006.1536.

[80] S. Lai, J. Harrington, Y. Xiang and K. Chow, "Accurate Analytical Perturbation Approach for Large Amplitude Vibration of Functionally Graded Beams," Int. J. Non-Linear Mech. (2011).

[60] M. Larfors, "Revisiting Toric $SU(3)$-Structures," Fortsch. Phys. **61** (2013) 1031 Fortsch. Phys. 61 (2013) 1031-1055, arXiv:1309.2953.

[73] O. Lechtenfeld, A. D. Popov and R. J. Szabo, "SU(3)-Equivariant Quiver Gauge Theories and Nonabelian Vortices," JHEP **0808** (2008) 093, arXiv:0806.2791.

[76] F. Lubbe, "Yang-Mills Solutions in Heterotic Flux Compactifications," Master's thesis, Leibniz Universität Hannover (2010).

[86] N. S. Manton and T. M. Samols, "Sphalerons On A Circle," Phys. Lett. B **207** (1988) 179.

[12] N. S. Manton and P. Sutcliffe, "Topological Solitons," Cambridge University Press (2004).

[66] B.-V. Matioc, "Qualitative Theorie Gewöhnlicher Differentialgleichungen," Lecture Notes, Leibniz Universität Hannover (2014).

[89] W. Merry, "Complex Manifolds," http://www.merry.io/resources/complexIII.pdf.

[45] F. Müller-Hoissen and R. Stuckl, "Coset Spaces and Ten-Dimensional Unified Theories," Class. Quant. Grav. **5** (1988) 27.

[42] M. Nakahara, "Geometry, Topology and Physics," Boca Raton, USA: Taylor & Francis (2003).

[90] A. Newlander and L. Nirenberg, "Complex Analytic Coordinates in Almost Complex Manifolds," Ann. of Math. (2) 65 (1957), 391404. MR 19,577a.

[57] C. Nölle, "Heterotic Supergravity on Manifolds with Killing Spinors," PhD thesis, Leibniz Universität Hannover (2012).

[46] C. Nölle, "Homogeneous Heterotic Supergravity Solutions with Linear Dilaton," J. Phys. A **45** (2012) 045402, arXiv:1011.2873.

[83] S. Parthasarathy and M. Lakshmanan, "On the Exact Solution of the Duffing Oscillator," J. Sound Vib. **137**, 523.

[19] A. D. Popov, "Antiselfdual Solutions of the Yang-Mills Equations in $4n$-Dimensions," Mod. Phys. Lett. A **7** (1992) 2077.

[75] T. Rahn, "Yang-Mills Configurations on Coset Spaces," Master's thesis, Leibniz Universität Hannover (2009).

[11] R. Rajaraman, "Solitons And Instantons. An Introduction To Solitons And Instantons In Quantum Field Theory," Amsterdam, Netherlands: North-Holland (1982).

[65] L. Rey-Bellet, "Differential Equations and Dynamical Systems," Classnotes for Math 645, University of Massachusetts, http://people.math.umass.edu/l̃r7q/ps_files/teaching/math645/ode.pdf.

[48] S. Salamon, "Riemannian Geometry and Holonomy Groups," Vol. 201 of Pitman Research Notes in Mathematics, Longman, Harlow (1989).

[49] F. Schulte-Hengesbach, "Half-Flat Structures on Lie Groups," PhD thesis, Universität Hamburg (2010).

[85] W. Schwalm, "Elliptic Functions sn, cn, dn, as Trigonometry," http://www.und.edu/instruct/schwalm/MAA_Presentation_10-02/handout.pdf.

[52] J. Sparks, "Sasaki-Einstein Manifolds," Surveys Diff. Geom. **16** (2011) 265, arXiv:arXiv:1004.2461.

[9] A. Strominger, "Superstrings with Torsion," Nucl. Phys. B **274** (1986) 253.

[56] D. Tong, "TASI Lectures on Solitons: Instantons, Monopoles, Vortices and Kinks," arXiv:hep-th/0509216.

[84] M. Tormählen, "Yang-Mills Solutions and Dyons on Cylinders over Coset Spaces with Sasakian Structure," arXiv:1412.7069.

[59] K. Uhlenbeck and S. T. Yau, "On the Existence of Hermitian-Yang-Mills Connections in Stable Vector Bundles," Comm. Pure Appl. Math., 39: S257-S293.

[88] S. Vandoren, "Lectures on Riemannian Geometry, Part II: Complex Manifolds."

[16] R. S. Ward, "Completely Solvable Gauge Field Equations in Dimension Greater Than Four," Nucl. Phys. B **236** (1984) 381.

[14] T. J. Willmore and N. J. Hitchin, "Global Riemannian Geometry," Ellis Horwood Ltd. (1984).